

Calculus of Variations for Differential Forms

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Résumé

Dans cette thèse, nous étudions le calcul des variations pour les formes différentielles.

Le première partie est dédiée au développement des outils des méthodes directes du calcul des variations pour résoudre des problèmes de minimisation de fonctionnelles d'une ou plusieurs variables de la forme

$$\int_{\Omega} f(d\omega), \quad \int_{\Omega} f(d\omega_1, \dots, d\omega_w), \quad \text{et} \quad \int_{\Omega} f(d\omega, \delta\omega).$$

Nous introduisons les notions de convexités appropriées à chaque cas, appelées *polyconvexité ext.*, *quasiconvexité ext.*, et *un-convexité ext.* pour des fonctionnelles de la forme $\int_{\Omega} f(d\omega)$, et la *polyconvexité ext. vectorielle*, la *quasiconvexité ext. vectorielle*, et la *un-convexité ext. vectorielle* pour des fonctionnelles de la forme $\int_{\Omega} f(d\omega_1, \dots, d\omega_m)$ ainsi que la *polyconvexité ext-int.*, la *quasiconvexité ext-int.* et la *un-convexité ext-int.* pour les fonctionnelles de la forme $\int_{\Omega} f(d\omega, \delta\omega)$. Nous étudions les liens et relations entre ces notions de convexité et leur homologues du cas classique du calcul des variations, c'est-à-dire, la polyconvexité, la quasiconvexité et la rang un convexité. Nous étudions également la semi-continuité inférieure et la continuité faible de ces fonctionnelles sur des espaces appropriés et nous nous occupons des problèmes de coercivité et obtenons des théorèmes d'existence à des problèmes de minimization de fonctionnelles d'une forme différentielle.

Dans la deuxième partie, nous étudions les problèmes aux limites pour des opérateurs de type Maxwell linéaires, semi-linéaires et quasi-linéaires pour des formes différentielles. Nous étudions l'existence et établissons la régularité intérieure ainsi que des estimations pour la régularité L^2 sur le bord pour l'opérateur de MAXwell linéaire

$$\delta(A(x)d\omega) = f$$

avec différentes conditions au bord ainsi que le système de type Hodge-Laplace associé

$$\delta(A(x)d\omega) + d\delta\omega = f,$$

avec les données au bord appropriées. Nous déduisons également sous la forme d'un corollaire l'existence et la régularité de solutions pour de systèmes du premier ordre de type div-rot. Nous déduisons également un résultat d'existence pour le problème au limites semi-linéaire

$$\begin{cases} \delta(A(x)(d\omega)) + f(\omega) = \lambda\omega \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Pour finir, nous discutons brièvement des résultats d'existence pour des opérateurs de Maxwell quasilineaires

$$\delta(A(x, d\omega)) = f,$$

avec différentes données au bord.

Mots-clés Calcul des variations, formes différentielles, quasiconvexité, problème de minimization, semicontinuité, opérateur de Maxwell.

Abstract

In this thesis we study calculus of variations for differential forms.

In the first part we develop the framework of direct methods of calculus of variations in the context of minimization problems for functionals of one or several differential forms of the type,

$$\int_{\Omega} f(d\omega), \quad \int_{\Omega} f(d\omega_1, \dots, d\omega_m) \quad \text{and} \quad \int_{\Omega} f(d\omega, \delta\omega).$$

We introduce the appropriate convexity notions in each case, called *ext. polyconvexity*, *ext. quasiconvexity* and *ext. one convexity* for functionals of the type $\int_{\Omega} f(d\omega)$, *vectorial ext. polyconvexity*, *vectorial ext. quasiconvexity* and *vectorial ext. one convexity* for functionals of the type $\int_{\Omega} f(d\omega_1, \dots, d\omega_m)$ and *ext-int. polyconvexity*, *ext-int. quasiconvexity* and *ext-int. one convexity* for functionals of the type $\int_{\Omega} f(d\omega, \delta\omega)$. We study their interrelationships and the connections of these convexity notions with the classical notion of polyconvexity, quasiconvexity and rank one convexity in classical vectorial calculus of variations. We also study weak lower semicontinuity and weak continuity of these functionals in appropriate spaces, address coercivity issues and obtain existence theorems for minimization problems for functionals of one differential forms.

In the second part we study different boundary value problems for linear, semilinear and quasilinear Maxwell type operator for differential forms. We study existence and derive interior regularity and L^2 boundary regularity estimates for the linear Maxwell operator

$$\delta(A(x)d\omega) = f$$

with different boundary conditions and the related Hodge Laplacian type system

$$\delta(A(x)d\omega) + d\delta\omega = f,$$

with appropriate boundary data. We also deduce, as a corollary, some existence and regularity results for div-curl type first order systems. We also deduce existence results for semilinear boundary value problems

$$\begin{cases} \delta(A(x)(d\omega)) + f(\omega) = \lambda\omega \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Lastly, we briefly discuss existence results for quasilinear Maxwell operator

$$\delta(A(x, d\omega)) = f,$$

with different boundary data.

Key words Calculus of variations, differential forms, quasiconvexity, minimization problem, semicontinuity, Maxwell operator.

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Contents

1	Introduction	5
1.1	Analysis with differential forms	5
1.2	Functional Analytic setting	6
1.3	Classical calculus of variations	7
1.4	Calculus of variations for differential forms	7
1.5	Functions of exterior derivative of a single differential form	8
1.6	Functions of exterior derivatives of several differential forms	13
1.7	Functions of exterior and interior derivative of a single differential form	17
1.8	Relationship with the classical calculus of variations	19
1.9	Maxwell operator	23
1.10	Organization	29
2	Differential forms	30
2.1	Introduction	30
2.2	Exterior forms	31
2.2.1	Definitions and main properties	31
2.2.2	Divisibility	33
2.3	Differential forms and their derivatives	34
2.4	Function Spaces of differential forms on \mathbb{R}^n	35
2.4.1	Partly Sobolev classes	35
2.4.2	Trace on partial Sobolev spaces	36
2.4.3	Gaffney inequality and Harmonic fields	40
2.5	Decomposition theorems and consequences	42
2.5.1	Hodge-Morrey decomposition	42
2.5.2	Classical boundary value problems for differential forms	44
2.5.3	Important Consequences	50
I	Direct Methods in Calculus of Variations for Differential Forms	54
3	Functionals depending on exterior derivative of a single differential form	56
3.1	Introduction	56
3.2	Notions of Convexity	57
3.2.1	Definitions	57
3.2.2	Preliminary lemmas	60

3.2.3	Main properties	65
3.3	The quasiaffine case	70
3.3.1	Some preliminary results	70
3.3.2	The characterization theorem	73
3.4	Examples	79
3.4.1	The quadratic case	79
3.4.2	Ext. quasiconvexity does not imply ext. polyconvexity	90
3.4.3	Ext one convexity does not imply ext quasiconvexity	92
3.4.4	Summary of implications and counter-implications	99
3.5	The ext convexity properties and the classical notions of convexity.	100
3.5.1	The projection maps	100
3.5.2	A crucial lemma	107
3.5.3	Equivalence theorem	118
3.6	Weak lower semicontinuity and existence theorems	120
3.6.1	Weak lower semicontinuity	120
3.6.2	Existence theorems in $W^{1,p}$	123
3.6.3	Existence theorems in $W^{d,p,q}$	128
4	Functionals depending on exterior derivatives of several differential forms	130
4.1	Introduction	130
4.2	Notions of Convexity	131
4.2.1	Definitions	131
4.2.2	Main Properties	133
4.3	Vectorially ext. quasiaffine functions	136
4.4	Weak lower semicontinuity	137
4.4.1	Necessary condition	137
4.4.2	Sufficient condition	142
4.5	Weak continuity	166
4.5.1	Necessary condition	166
4.5.2	Sufficient condition	167
5	Other types of functionals	170
5.1	Introduction	170
5.2	Notions of Convexity	171
5.2.1	Definitions	171
5.2.2	Main properties	172
5.3	The quasiaffine case	173
5.3.1	Some preliminary results	173
5.3.2	The characterization theorem	174
5.4	The ext-int convexity properties and the classical notions of convexity.	179
5.4.1	The projection maps	179
5.5	Existence theorems for minimization problems	181
5.5.1	Existence theorems without lower order terms	181

5.5.2	Existence theorems with lower order terms	182
II	Some Boundary value problems for Differential Forms	185
6	Maxwell operator for k-forms: Linear theory	189
6.1	Introduction	189
6.2	Existence of weak solutions	191
6.2.1	Existence in $W_{\delta,T}^{d,2}$	192
6.2.2	Existence in $W_T^{d,2}$	197
6.2.3	Existence theorems	198
6.3	Interior regularity of weak solutions	200
6.4	Regularity up to the boundary	205
6.5	Main theorems	225
6.6	Applications of Linear theory	230
6.6.1	Div-Curl type first order linear system	230
6.6.2	Hodge Laplacian type elliptic system	232
7	Maxwell operator for k-forms: Nonlinear Case	235
7.1	Introduction	235
7.2	Semilinear theory: Coercive case	237
7.2.1	Existence of weak solutions	237
7.3	Semilinear theory: Strongly Indefinite case	242
7.3.1	Abstract critical point theory for indefinite functionals	242
7.3.2	Existence of weak solutions	250
7.4	Quasilinear theory	262
7.4.1	Existence of weak solutions	262
7.4.2	Main theorems	265
7.4.3	Remark about regularity	267
	Appendix A Notations	268
	Appendix B Function Spaces of Differential Forms	273
B.1	Usual Function Spaces	273
B.2	Special function spaces	276

Chapter 1

Introduction

1.1 *Analysis with differential forms*

Differential forms are among the fundamental objects in geometry, topology and global analysis. All the familiar operators from vector calculus like gradient, curl and divergence and the related identities are best expressed, in a crisp manner, in the language of differential forms. Also differential forms, as mathematical objects, are independent of the coordinate system we choose to describe them in local co-ordinates. This makes them handy in manifolds, where they carry real geometric meaning on one hand and on the other hand allow us to manipulate them using any local coordinate system we deem convenient. The alternating structure of the exterior algebra is also extremely rich in its own right and this algebraic structure also behaves unexpectedly well with respect to the topology of componentwise weak convergence, as we shall make explicit later in this thesis.

Partial differential equations, on the other hand, have always been the heart of analysis. Time and again in the rich history of partial differential equations, it has been observed that those equations or systems of equations which have a variational structure, i.e appear as Euler-Lagrange equations of functionals, typically integral functionals, are by far the most important subclass of problems. There are several reasons for this. The most primary reason for this is that they tend to be ubiquitous in mathematics and even in other branches of science. On the other hand, their variational structure makes them amenable to a variety of techniques which are inapplicable in the non-variational case. One such example is the extremely powerful techniques of the so called *direct methods*, where one works directly with the functional instead of the equation to prove existence. Such methods can be broadly classified into two rather distinct classes:

- Direct Minimization,
- Critical Point methods.

If the functional is bounded below then the strategy is to solve a minimization problem. If we can prove the existence of a minimizer in a suitable function space, then this minimizer will solve the Euler-Lagrange equation at least in some weak sense. Of course, the strategy is the same if the functional is only bounded above, in which case we just consider the negative of the functional. But when the functional is unbounded both above and below, of course we can have

no hope of solving the minimization problem and we look for other methods to look for critical points of the functional, typically for saddle-type critical points as opposed to local minima type critical points in the minimization case.

However, in spite of the fact that differential forms are well-known and widely useful geometric objects and variational methods are, by now, quite well developed for equations involving scalar valued and even vector valued unknown functions, variational problems for differential forms have not attracted the same amount of attention. Non-variational problems for differential forms have been studied even less. Nowadays, there has been a growing interest in these problems coming from branches like quasiregular mappings, gauge theory, harmonic maps between manifolds, pullback equations for differential forms, optimal transports etc, just to name a few. But still, some areas received surprisingly little attention till date. This thesis is a contribution to a number of such areas.

One such area is the direct minimization techniques for integral functionals with possibly non-convex integrands. To the best of our knowledge, this has not been studied systematically even in the simplest case of smooth, bounded subsets of \mathbb{R}^n . The first part of this thesis deals with the situation. In this part, our main interest is to develop a framework for solving a class of minimization problems involving differential forms, the simplest of which typically has the form,

$$m := \inf \left\{ \int_{\Omega} f(d\omega) : \omega \in X(\Omega) \right\},$$

where Ω is an open, bounded subset of \mathbb{R}^n with smooth enough boundary, $\omega : \Omega \subset \mathbb{R}^n \rightarrow \Lambda^k(\mathbb{R}^n)$ is a differential k -form on Ω , $f : \Lambda^{k+1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ is a given continuous function and $X(\Omega)$ is a function space of differential forms on Ω . The principal question related to this problem is the existence of a minimizer in a suitable space $X(\Omega)$. But before discussing the problems we shall treat in details, first a few remarks about the functional analytic setting are in order.

1.2 Functional Analytic setting

Nonsmooth differential forms It is well-known in the analysis of partial differential equations and calculus of variations that the Sobolev spaces are particularly well adapted to existence problems. It is much easier, in general to obtain existence results in Sobolev spaces than in some other space of more regular functions. However, differential forms are generally defined on a smooth manifold, using the smooth structure, i.e the smooth charts and atlases and are therefore not well suited for our purpose. For this reason, we shall define and work with non-smooth differential forms, whose components are measurable functions and not necessarily smooth.

Sobolev spaces and partial Sobolev spaces For essentially the same reason as above, we need to define Sobolev spaces of differential forms. But apart from the usual $W^{1,p}$ spaces, we shall also work with the so-called partial Sobolev spaces. These are spaces of L^p forms for which we require that some combination of derivatives (in contrast to all of the derivatives in the $W^{1,p}$ case) are in L^p . Also, in contrast to the standard Sobolev spaces, they do not have a

well-defined Trace map to the boundary. However, the spaces we shall work with, for example $W^{d,p}$, i.e the space of forms ω for which $\omega, d\omega \in L^p$, does have partial Trace maps. In the case of $W^{d,p}$ spaces, only the ‘tangential trace’ to the boundary can be defined (cf. Chapter 2 for more on this).

1.3 Classical calculus of variations

Now we discuss the abstract framework of classical calculus of variations. When $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a vector valued function, the minimization problem

$$m := \inf \left\{ \int_{\Omega} f(\nabla u) : u \in X(\Omega) \right\},$$

is well studied and forms the subject matter of the so-called direct methods in calculus of variations. In this case, the spaces $X(\Omega)$ are generally Sobolev spaces of $W^{1,p}$ type, often with prescribed boundary values. The conditions on the integrand f , which guarantees the existence of minimizer are well-known. These conditions can typically be classified into two types: *convexity conditions* and *growth conditions* or *coercivity conditions*.

The functional analytic framework is rather simple. The growth condition ensures coercivity, i.e they ensure that when the the value of the integral decreases to the infimum value, the Sobolev norm of the minimizing sequence remains bounded. This implies that the minimizing sequence is bounded in a Sobolev space. Hence up to a subsequence, these sequences converge, weakly to a limit. The convexity condition essentially ensures the sequential lower semicontinuity of the functional with respect to the weak topology. This in turn implies that the weak limit of the minimizing sequence is itself a minimizer. The subject matter of direct methods in classical calculus of variations is therefore finding fairly general convexity and growth conditions. Our goal is to build a similar framework for functionals of differential forms. The growth conditions we use is essentially the same as the ones in classical calculus of variations. So we focus mainly on the convexity conditions. The relevant convexity conditions in the classical calculus of variations, apart from convexity, are called rank one convexity, quasiconvexity and polyconvexity. Our aim is to find analogous conditions in the case of differential forms.

1.4 Calculus of variations for differential forms

With the understanding that differential form always mean their nonsmooth cousins and the basic spaces in which to prove existence are different partial and standard Sobolev spaces, we can now focus on our problems in more detail. The domain Ω for us is always an open, bounded subset of \mathbb{R}^n with smooth enough boundary. Often we impose topological restriction on the domain also. We investigate the existence of minimizers for minimization problems for the following type of functionals:

$$\int_{\Omega} f(d\omega), \tag{1.1}$$

$$\int_{\Omega} f(d\omega_1, \dots, d\omega_m), \tag{1.2}$$

and

$$\int_{\Omega} f(d\omega, \delta\omega). \quad (1.3)$$

Analysis of (1.1) can be seen as an unified way of dealing with minimization of functionals of the form

- $\int_{\Omega} f(\nabla\omega), \quad (k = 1)$
- $\int_{\Omega} f(\text{curl}\omega), \quad (k = 2)$
- $\int_{\Omega} f(\text{div}\omega) \quad (k = n)$

etc. (1.2) is a much more general version of these, involving multiple unknown differential forms, which also generalizes classical calculus of variations. Whereas for $k = 1$ and $n = 3$, (1.3) reduces to the form $\int_{\Omega} f(\text{curl}\omega, \text{div}\omega)$. The main focus is primarily on finding the correct notions of convexity.

1.5 Functions of exterior derivative of a single differential form

For functionals of the form (1.1), we introduce the appropriate notions of convexity which are named, for want of a better terminology, *ext. one convexity*, *ext. quasiconvexity* and *ext. polyconvexity*, which plays the analogous roles played by rank one convexity, quasiconvexity and polyconvexity respectively, in classical calculus of variations. A function $f : \Lambda^k \rightarrow \mathbb{R}$ is called

- *ext. one convex* if it satisfies,

$$f(t\xi + (1-t)\eta) \leq tf(\xi) + (1-t)f(\eta),$$

for every $\xi, \eta \in \Lambda^k$ such that there exists $a \in \Lambda^1, b \in \Lambda^{k-1}$ with $\xi - \eta = a \wedge b$. This is just convexity in the direction of the 1-divisible forms, i.e k -forms which can be written as a wedge product of a $k - 1$ form and an 1-form.

- *ext. quasiconvex* if it satisfies,

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{meas } \Omega$$

for every bounded open set $\Omega \subset \mathbb{R}^n$, for every $\xi \in \Lambda^k$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$. This is just the usual quasiconvexity inequality, except that we have the operator d in place of the gradient.

- *ext. polyconvex*, if there exists a convex function

$$F : \Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k} \rightarrow \mathbb{R}$$

such that

$$f(\xi) = F\left(\xi, \xi^2, \dots, \xi^{[n/k]}\right),$$

where

$$\xi^m := \underbrace{\xi \wedge \dots \wedge \xi}_{m \text{ times}}.$$

This just means that an ext. polyconvex function is a convex function of all the wedge powers.

The corresponding notions for the case with δ , the codifferential operators, which is the formal adjoint of the operator d , are called *int. one convexity*, *int. quasiconvexity* and *int. polyconvexity*.

The definition of *ext. quasiconvexity* is reminiscent of the definitions of A -quasiconvexity and $A - B$ -quasiconvexity in classical calculus of variations, introduced by Dacorogna in [22] and [23]. The definition of *ext. one convexity* is basically the convexity in the direction of the ‘wave-cone’, a concept introduced by Tartar in [67], of the operator d . The definition of *ext. polyconvexity*, however depends on the characterization of ext.-quasiaffine functions, which has been obtained here for the first time. We then proceed to analyze the relationships between these notions of convexity.

Ext. Quasiaffine functions The first crucial theorem is Theorem 3.20, which characterizes all ext. quasiaffine functions. The theorem shows that for any $f : \Lambda^k \rightarrow \mathbb{R}$,

$$f \text{ ext. polyaffine} \Leftrightarrow f \text{ ext. quasiaffine} \Leftrightarrow f \text{ ext. one affine}$$

and any such function f is necessarily of the form

$$f(\xi) = \sum_{s=0}^{[n/k]} \langle c_s; \xi^s \rangle \quad \text{for any } \xi \in \Lambda^k$$

for some forms $c_s \in \Lambda^{ks}$, $0 \leq s \leq [n/k]$, where $\xi^0 \in \Lambda^0$ is defined as the constant function 1 by convention. It basically says that all the convexity notions coincide at the level of affinity and that any ext. quasiaffine function is a linear combination (up to a constant) of the nontrivial wedge powers. For example, this shows that for $k = 2, n = 4$, the nonlinear function

$$f(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle \quad \text{for any } \xi \in \Lambda^2(\mathbb{R}^4)$$

is ext. quasiaffine and any non-affine ext. quasiaffine function $g : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$ is of the form $cf(\xi)$, for some non-zero real number c , modulo affine functions of ξ . This result is analogous to the characterization theorem for quasiaffine functions in the classical case, established by Ball in [4] (see Theorem 5.20 in Dacorogna[25]).

Already this theorem shows several peculiarities of the algebraic structure of the exterior forms. Since whenever k is an odd integer, $\xi \wedge \xi = 0$ for every $\xi \in \Lambda^k(\mathbb{R}^n)$ in any dimension n , this implies that if k is odd, all ext. quasiaffine functions are actually affine in any dimension n . This in turn implies the striking result that ext. polyconvexity is equivalent to convexity, as soon as k is odd.

Relationship between notions of convexity Next we analyze the interrelationship between *ext. one convexity*, *ext. quasiconvexity* and *ext. polyconvexity* in great detail for any $1 \leq k \leq n$ in any dimension n . The results obtained are summarized in Theorem 3.37. Before proceeding, recall that the case $k = 1$ is the classical case of the gradient of a scalar function, i.e the ‘scalar case’ of classical calculus of variations.

The theorem asserts that if $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$,

(i) The following implications then hold

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex} \Rightarrow f \text{ ext. one convex.}$$

(ii) If $k = 1, n - 1, n$ or $k = n - 2$ is odd, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Moreover if k is odd or $2k > n$, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iii) If either $k = 2$ and $n \geq 4$ or $3 \leq k \leq n - 3$ or $k = n - 2 \geq 4$ is even, then

$$f \text{ ext. polyconvex} \xRightarrow{\quad} f \text{ ext. quasiconvex}$$

while if $2 \leq k \leq n - 3$ (and thus $n \geq k + 3 \geq 5$), then

$$f \text{ ext. quasiconvex} \xRightarrow{\quad} f \text{ ext. one convex.}$$

The last counter implication is reminiscent of the counter example of Šverák in the classical calculus of variations (see [65]), with an additional algebraic construction, which is quite involved.

This yields a complete picture of the implications and counter implications, except the counter implication

$$f \text{ ext. quasiconvex} \not\Rightarrow f \text{ ext. one convex}$$

for the case when $k = n - 2 \geq 2$ is even. This means the critical dimensions for which we can not settle the counter-implication for a k -form is $k + 2$, when k is even.

Quadratic case Quadratic functions, i.e functions of the form

$$f(\xi) = \langle M\xi; \xi \rangle \quad \text{for any } \xi \in \Lambda^k,$$

for some symmetric linear operator $M : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^n)$, form an important subclass since in this case the Euler-Lagrange equations for the minimization problem are linear. Hence we also analyze this special case thoroughly and the results obtained are summarized in Theorem 3.30, which states:

For quadratic functions,

(i) The following equivalence holds in all cases

$$f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(ii) Let $k = 2$. If $n = 2$ or $n = 3$, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

If $n = 4$, then

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \nLeftrightarrow \end{matrix} f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex}$$

while if $n \geq 6$, then

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \nLeftrightarrow \end{matrix} f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \nLeftrightarrow \end{matrix} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(iii) If k is odd or if $2k > n$, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iv) If k is even and $2k \leq n$, then

$$f \text{ convex} \begin{matrix} \Rightarrow \\ \nLeftrightarrow \end{matrix} f \text{ ext. polyconvex.}$$

(v) If either $3 \leq k \leq n - 3$ or $k = n - 2 \geq 4$ is even, then

$$f \text{ ext. polyconvex} \begin{matrix} \Rightarrow \\ \nLeftrightarrow \end{matrix} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

As can be seen from above, the picture is also complete in this case except that the equivalence between polyconvexity and quasiconvexity remains open for $k = 2$ and $n = 5$.

The analogy of these results with the classical case of the gradient of a vector-valued function $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$ is also interesting. The analogue to conclusion (i) in the classical vectorial calculus of variations, i.e the result that for quadratic functions $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$,

$$f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex ,}$$

was first proved by Van Hove ([72],[73]), though it was implicitly known earlier.

For quadratic functions $f : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$, the equivalence in the classical case,

$$f \text{ polyconvex} \Leftrightarrow f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex,}$$

has a long history involving contributions by Albert [3], Hestenes-McShane [37], McShane [46], Marcellini [47], Reid [56], Serre [59], Terpstra [69] and Uhlig [71]. For $k = 2$ and $n = 4$, the

proof of the equivalence

$$f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex},$$

is reminiscent of the ideas in Hestenes-McShane [37], Marcellini [47] and Uhlig [71].

If $N, n \geq 3$ in the classical vectorial calculus of variations, then for quadratic functions, in general

$$f \text{ polyconvex} \xrightarrow{\neq} f \text{ rank one convex}.$$

The counter example was given by Terpstra [69] and later by Serre [59] (see also Ball [5], Davit-Milton [36] provides another recent counterexample). The proof of the counter implication

$$f \text{ ext. polyconvex} \xrightarrow{\neq} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex},$$

for $k = 2$ and $n \geq 6$ is inspired by ideas used in constructing the abovementioned counterexample in Serre [59] and Terpstra [69].

Existence theorems After the analysis of convexity notions, we turn our attention to existence theorems for minimization problems. There are two aspects of the difficulties involved. The first one is the weak lower semicontinuity and the second one is the coercivity of the functional (1.1). Ext. quasiconvexity of f is enough to ensure weak lower semicontinuity of the functional in the appropriate space with the usual growth assumptions on f . However, there is already a striking difference from the classical case. From the point of view of weak lower semicontinuity, the appropriate space is $W^{d,p}$, not $W^{1,p}$. This already poses the significant difficulty that when f is ext. quasiconvex, although the functionals $\int_{\Omega} f(d\omega)$ and $\int_{\Omega} f(x, d\omega)$ are semicontinuous in $W^{d,p}$, the functionals of the form $\int_{\Omega} f(x, \omega, d\omega)$, i.e functionals with explicit dependence on ω are generally not weakly lower semicontinuous on $W^{d,p}$. For example, the functional

$$I(\omega) = \frac{1}{p} \int_{\Omega} |d\omega|^p - \frac{1}{p} \int_{\Omega} |\omega|^p$$

is not weakly lower semicontinuous on $W^{d,p}(\Omega; \Lambda^k)$ as soon as $k \geq 2$. However, when $k = 1$, this functional is

$$I(\omega) = \frac{1}{p} \int_{\Omega} |\nabla \omega|^p - \frac{1}{p} \int_{\Omega} |\omega|^p,$$

which is weakly lower semicontinuous on $W^{d,p}(\Omega; \Lambda^1)$. Note also that for $k = 1$, the two spaces coincide, i.e $W^{d,p}(\Omega; \Lambda^1) = W^{1,p}(\Omega; \Lambda^1)$. On the other hand, though all these functionals are weakly lower semicontinuous on $W^{1,p}$, the functionals are not a priori coercive on $W^{1,p}$ with the usual growth assumptions, since those assumptions only imply that for any minimizing sequence ω_ν , the sequence $d\omega_\nu$ is uniformly bounded in L^p , but $\nabla \omega_\nu$ need not be.

However, for functionals of the form $\int_{\Omega} f(d\omega)$ and $\int_{\Omega} f(x, d\omega)$, these difficulties can be circumvented by solving certain type of boundary value problems involving differential forms. In fact, in these cases it can be shown that the existence for the minimizer on $W^{1,p}$ can be derived from the existence of minimizer of the same functional on another subspace of $W^{d,p}$,

on which the functional is both coercive and weakly lower semicontinuous. This is achieved in Theorem 3.64, which in effect proves the existence of a minimizer, under the assumption of ext. quasiconvexity and usual growth assumptions on f , for the following minimization problem

$$\inf \left\{ \int_{\Omega} f(x, d\omega) : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m.$$

Of course, by Hodge duality, all the preceding discussion is also true for functionals of the form

$$\int_{\Omega} f(\delta\omega), \int_{\Omega} f(x, \delta\omega) \text{ and } \int_{\Omega} f(x, \omega, \delta\omega)$$

as well if we replace $W^{d,p}$ by $W^{\delta,p}$. Theorem 3.67 gives the existence result for the corresponding problem for $\int_{\Omega} f(x, \delta\omega)$. These two theorems also imply that addition of terms which are linear in ω with special structure still enables us to solve the minimization problem. We also show in Theorem 3.69 that when the explicit dependence on ω is an additive term which is convex, coercive and nonnegative, i.e the functional is of the form

$$\int_{\Omega} [f(x, d\omega) + g(x, \omega)],$$

with g being nonnegative and convex and coercive with respect to ω variable, existence of minimizer can be obtained in a subspace of $W^{d,p}$, which however, is in general larger than $W^{1,p}$.

1.6 Functions of exterior derivatives of several differential forms

After analyzing the situation for the functional (1.1), we turn our attention to functionals of the form (1.2). Our first priority is, once again, to figure out the correct convexity notions. The appropriate notions, called *vectorial ext. one convexity*, *vectorial ext. quasiconvexity* and *vectorial ext. polyconvexity* are introduced.

A function $f : \Lambda^{k_1} \times \dots \times \Lambda^{k_m} \rightarrow \mathbb{R}$ is called

- *vectorially ext. one convex* if it satisfies,

$$f(t\xi_1 + (1-t)\eta_1, \dots, t\xi_m + (1-t)\eta_m) \leq tf(\xi_1, \dots, \xi_m) + (1-t)f(\eta_1, \dots, \eta_m),$$

for every collection $\xi_i, \eta_i \in \Lambda^{k_i}$ such that there exists $a \in \Lambda^1, b_i \in \Lambda^{k_i-1}$ with $\xi_i - \eta_i = a \wedge b_i$, for all $1 \leq i \leq m$.

- *vectorially ext. quasiconvex* if it satisfies,

$$\frac{1}{\text{meas}(\Omega)} \int_{\Omega} f(\xi_1 + d\omega_1(x), \xi_2 + d\omega_2(x), \dots, \xi_m + d\omega_m(x)) \geq f(\xi_1, \xi_2, \dots, \xi_m)$$

for every bounded open set Ω , for every collection of $\xi_i \in \Lambda^{k_i}$, $1 \leq i \leq m$ and for every $\omega_i \in W_0^{1,\infty}(\Omega; \Lambda^{k_i-1})$, $1 \leq i \leq m$.

- *vectorially ext. polyconvex*, if there exists a convex function such that

$$f(\xi_1, \dots, \xi_m) = F(T_1(\xi_1, \dots, \xi_m), \dots, T_N(\xi_1, \dots, \xi_m)),$$

where

$$N := \left\lceil \frac{n}{\min_{1 \leq i \leq m} k_i} \right\rceil$$

and for every $1 \leq r \leq N$, $T_r(\xi_1, \dots, \xi_m)$ denotes the vectors with components $\xi_1^{\alpha_1} \wedge \dots \wedge \xi_m^{\alpha_m}$, where the nonnegative integers α_i s vary over all possible choices such that $\sum_{i=1}^m \alpha_i = r$. This just means that a *vectorially ext. polyconvex* function is a convex function of all the possible wedge products of the arguments of the function, including repeated products.

Once again, the definition of *vectorial ext. one convexity* is basically the convexity in the direction of the ‘wave-cone’ in this case (see Dacorogna [23] and references therein). The definition of *vectorial ext. quasiconvexity*, however, already appeared in Iwaniec-Lutoborski [39], which the authors simply called *quasiconvexity*. The same article also defines a notion of *polyconvexity*, which coincides with *vectorial ext. polyconvexity* if all the k_i s are odd integers, but in general is a strict subclass of *vectorially ext. polyconvex* functions. For example, the function $f_1 : \Lambda^{k_1} \times \Lambda^{k_2} \rightarrow \mathbb{R}$ given by,

$$f_1(\xi_1, \xi_2) = \langle c; \xi_1 \wedge \xi_2 \rangle \quad \text{for every } \xi_1 \in \Lambda^{k_1}, \xi_2 \in \Lambda^{k_2}$$

where $c \in \Lambda^{(k_1+k_2)}$ is constant, is *polyaffine* in the sense of Iwaniec-Lutoborski [39] and also *vectorially ext. polyaffine*. However, the function $f_2 : \Lambda^{k_1} \times \Lambda^{k_2} \rightarrow \mathbb{R}$ given by,

$$f_2(\xi_1, \xi_2) = \langle c; \xi_1 \wedge \xi_1 \rangle \quad \text{for every } \xi_1 \in \Lambda^{k_1}, \xi_2 \in \Lambda^{k_2}$$

where $c \in \Lambda^{2k_1}$ is constant, is *vectorially ext. polyaffine*, but not *polyaffine* in the sense of Iwaniec-Lutoborski [39]. Note also that it is easy to see, by integrating by parts that both f_1 and f_2 are *vectorially ext. quasilinear* and hence are also *quasilinear* in the sense of Iwaniec-Lutoborski [39]. Also, when $m = 1$, i.e there is only one differential form, reducing the problem to the form (1.1), their definition of *polyconvexity* coincide with usual convexity. On the other hand, when $m = 1$, *vectorial ext. polyconvexity* reduces to *ext. polyconvexity*, which is much weaker than convexity.

We do not pursue the interrelationship between the notions of convexity in great detail, though we believe that it can indeed prove to be rewarding. We of course obtain the basic relationship which states,

$$\begin{aligned} f \text{ convex} &\Rightarrow f \text{ vectorially ext. polyconvex} \Rightarrow f \text{ vectorially ext. quasiconvex} \\ &\Rightarrow f \text{ vectorially ext. one convex.} \end{aligned}$$

Since we have already studied the counter-implication in great detail for the simpler case of single forms, instead of pursuing such a course, we move on to the characterization of *vectorially*

ext. quasilinear functions.

Vectorially Ext. Quasilinear functions The crucial theorem is Theorem 4.12, which once again establishes the expected fact that

$$f \text{ vectorially ext. polyaffine} \Leftrightarrow f \text{ vectorially ext. quasilinear} \Leftrightarrow f \text{ vectorially ext. one affine}$$

and any such function f is necessarily a linear combination of all possible nontrivial wedge products of the arguments of f . This result, although a natural development from the perspective of our program so far, is actually powerful enough to yield the classical result about the quasilinear functions (cf. theorem 5.20 in Dacorogna [25]) as a special case. In fact, this result points towards a natural framework to look at classical calculus of variations. Classical quasilinear functions are linear combinations of determinants and adjugates because they are precisely the wedge products when one considers each row of the matrix as a 1-form.

Semicontinuity

Motivated by the last observation, we turn towards tackling one of the central problems in all of calculus of variations, namely weak lower semicontinuity and ask whether the setting of several differential form is the more natural setting to study the semicontinuity problem. We obtain an answer in the affirmative but the results at the same time shows the special feature of the gradient case which is absent for the this general setting. If the functional is of the form

$$\int_{\Omega} f(x, d\omega_1, \dots, d\omega_m),$$

i.e they do not have any explicit dependence on $\omega_1, \dots, \omega_m$, the semicontinuity result is given in Theorem 4.25.

Let k_1, \dots, k_m be m integers where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. Let p_1, \dots, p_m be extended real numbers such that $1 \leq p_i \leq \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \Omega \times \Lambda^{k_1} \times \dots \times \Lambda^{k_m} \rightarrow \mathbb{R}$ be a Carathéodory function, satisfying the growth condition, for almost every $x \in \Omega$ and for every collection $(\xi_1, \dots, \xi_m) \in \Lambda^{k_1} \times \dots \times \Lambda^{k_m}$,

$$-\beta(x) - \sum_{i=1}^m G_i^l(\xi_i) \leq f(x, \xi_1, \dots, \xi_m) \leq \beta(x) + \sum_{i=1}^m G_i^u(\xi_i),$$

where $\beta \in L^1(\Omega)$ is nonnegative and the functions G_i^l s in the lower bound and the functions G_i^u s in the upper bound has the following form:

- If $p_i = 1$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$G_i^l(\xi_i) = \alpha_i |\xi_i|^{q_i}$$

and

$$G_i^u(\xi_i) = g_i(x) |\xi_i|^{p_i},$$

for some $1 \leq q_i < p_i$ and for some constant $\alpha_i \geq 0$ and some non-negative measurable function g_i .

- If $p_i = \infty$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \eta_i(|\xi_i|).$$

for some nonnegative, continuous, increasing function η_i .

Also let $(\xi_1, \dots, \xi_m) \mapsto f(x, \xi_1, \dots, \xi_m)$ is vectorially ext. quasiconvex for a.e $x \in \Omega$. Let $\{\omega_i^\nu\}$ be sequences such that for every $1 \leq i \leq m$, we have,

$$\omega_i^\nu \rightharpoonup \omega_i \quad \text{in } W^{d, p_i} \quad (\overset{*}{\rightharpoonup} \text{ if } p_i = \infty),$$

for some $\omega_i \in W^{d, p_i}(\Omega)$, then Theorem 4.25 says that we have,

$$\liminf_{\nu \rightarrow \infty} \int_{\Omega} f(x, d\omega_i^\nu, \dots, d\omega_m^\nu) dx \geq \int_{\Omega} f(x, d\omega_1, \dots, d\omega_m) dx.$$

In other words, the theorem says that assuming a growth condition on $f = f(x, \xi_1, \dots, \xi_m)$, which is basically just the sum of usual power type growth conditions on each argument of f , the functional is weakly lower semicontinuous in the product space $W^{d, p_1} \times \dots \times W^{d, p_m}$, where p_i are the powers that appear in the growth condition for each argument ξ_i , if $(\xi_1, \dots, \xi_m) \mapsto f(x, \xi_1, \dots, \xi_m)$ is vectorially ext. quasiconvex for a.e $x \in \Omega$. Note also that these exponents p_i s are allowed to be different from one another and are allowed to take any value between $1 \leq p_i \leq \infty$, i.e both 1 and ∞ is included.

The proof of this semicontinuity result uses ideas which are reminiscent of the proof of the semicontinuity results in classical case by Acerbi-Fusco [1] and Marcellini [48], which are used in combination with a classical lemma on equiintegrability of Fonseca-Muller-Pedregal [30] and Kristensen [43], along with lemma 4.17, which is a generalization of a classical result relating quasiconvexity with $W^{1, p}$ -quasiconvexity in the classical case by Ball-Murat [8]. The other crucial ingredient is proposition 4.19, which generalizes the classical Lipschitz inequality for separately convex functions with growth assumptions (cf. Proposition 2.32 in Dacorogna [25]) in Fusco [31], Marcellini [48], Morrey [53]. We remark that all our results related to sufficiency of vectorial ext. quasiconvexity for weak lower semicontinuity, e.g. lemma 4.21, theorem 4.22 and theorem 4.25 can also be proved in a different manner by introducing Young measures and utilizing the blow-up argument of Fonseca-Muller [29]. However, in this thesis we refrain from introducing Young measures.

However, if the functional has explicit dependence on $\omega_1, \dots, \omega_m$, i.e it is of the form

$$\int_{\Omega} f(x, \omega_1, \dots, \omega_m, d\omega_1, \dots, d\omega_m),$$

then the functional is not necessarily weakly lower semicontinuous in $W^{d,p_1} \times \dots \times W^{d,p_m}$ with the usual growth assumptions. For example, even for the simplest case of $m = 1$, the functional we mentioned before, i.e

$$I(\omega) = \frac{1}{p} \int_{\Omega} |d\omega|^p - \frac{1}{p} \int_{\Omega} |\omega|^p$$

is a counter example if $k > 1$. However it can be shown that the functional of the form

$$\int_{\Omega} f(x, \omega_1, \dots, \omega_m, d\omega_1, \dots, d\omega_m),$$

is nonetheless, weakly lower semicontinuous in $W^{1,p_1} \times \dots \times W^{1,p_m}$. The real issue here is that the L^p norm of $d\omega$ can not control the L^p norm of ω , i.e the unavailability of Sobolev-Poincaré inequalities in $W^{d,p}$ spaces. Theorem 4.14 proves the necessity of vectorial ext. quasiconvexity for weak lower semicontinuity.

Weak continuity and compensated compactness The semicontinuity results and the characterization of vectorially ext. quasilinear functions paves the way to inspect closely the relationship between weak convergence and wedge products. It is well known that nonlinear terms, in general, do not behave well with respect to weak convergence, i.e in more precise terms, a general nonlinear function which is continuous need not be continuous with respect to the weak topology. However, for weakly convergent sequences for which there is an uniform bound on some combination of derivatives, there can be nonlinear functions which are still ‘weakly continuous’ on such sequences, i.e the the image sequence converges, in some weak topology, to the image of the weak limit. This class of nonlinear functions, called ‘*Null Lagrangians*’, of course depend on the combination of derivatives for which we can deduce the uniform bounds. This, in essence, is the philosophy of the theory of compensated compactness and is explained in Tartar [67].

We shall restrict our attention to the case of the exterior derivative, i.e we shall try to find nonlinear functions which are ‘weakly continuous’ with respect to sequences with uniformly bounded exterior derivative. This has been investigated first in Robin-Rogers-Temple [57]. Theorem 4.34 proves the weak continuity of wedge products. The borderline case, i.e when the wedge products are only L^1 , we have used the result presented in Robin-Rogers-Temple [57], but we supply a new proof based on the semicontinuity theorems for the other cases. Theorem 4.31 answers the question posed in the same paper in the affirmative, i.e it proves that any such ‘weakly continuous’ functions must be a linear combination of wedge products.

1.7 Functions of exterior and interior derivative of a single differential form

Functionals of the form (1.3) present fewer challenges than what we might expect. By the classical Gaffney inequality, for differential forms satisfying certain boundary conditions, if we

can control the L^p norm of both $d\omega$ and $\delta\omega$, then we can control the L^p norm of $\nabla\omega$, i.e the norms of all the first order derivatives can be controlled. So there is no the lack of coercivity and Sobolev-Poincaré type inequalities are also available, making the analysis simpler in this case in this respect. Figuring out the appropriate convexity conditions is still a reasonable goal and we introduce the notions, which we called, again for want of anything better, *ext-int. one convexity*, *ext-int. quasiconvexity* and *ext-int. polyconvexity*. We call a function $f : \Lambda^{k+1} \times \Lambda^{k-1} \rightarrow \mathbb{R}$ is called

- *ext-int. one convex* if it satisfies,

$$f(t\xi_1 + (1-t)\xi_2, t\eta_1 + (1-t)\eta_2) \leq tf(\xi_1, \eta_1) + (1-t)f(\xi_2, \eta_2),$$

for every $\xi_1, \xi_2 \in \Lambda^{k+1}$, $\eta_1, \eta_2 \in \Lambda^{k-1}$ such that there exists $a \in \Lambda^1, b \in \Lambda^k$ with $\xi_1 - \xi_2 = a \wedge b$ and $\eta_1 - \eta_2 = a \lrcorner b$.

- *ext-int. quasiconvex* if it satisfies,

$$\int_{\Omega} f(\xi + d\omega, \eta + \delta\omega) \geq f(\xi, \eta) \text{ meas } \Omega$$

for every $\xi \in \Lambda^{k+1}, \eta \in \Lambda^{k-1}$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$.

- *ext-int. polyconvex*, if there exists a convex function

$$F : \Lambda^{k+1} \times \Lambda^{2(k+1)} \times \dots \times \Lambda^{\left[\frac{n}{k+1}\right](k+1)} \times \Lambda^{n-k+1} \times \Lambda^{2(n-k+1)} \times \dots \times \Lambda^{\left[\frac{n}{n-k+1}\right](n-k+1)} \rightarrow \mathbb{R}$$

such that

$$f(\xi, \eta) = F\left(\xi, \xi^2, \dots, \xi^{\left[\frac{n}{k+1}\right]}, *\eta, (*\eta)^2, \dots, (*\eta)^{\left[\frac{n}{n-k+1}\right]}\right).$$

Once again, the definition of *ext-int. quasiconvexity* is along the lines of A -*quasiconvexity* and $A-B$ -*quasiconvexity*. The definition of *ext-int. one convexity* is just convexity in the directions of the 'wave cone' of the differential operator (d, δ) , acting componentwise.

We do not study the interrelationships in great detail here either. We deduce the basic result

$$f \text{ convex} \Rightarrow f \text{ ext-int. polyconvex} \Rightarrow f \text{ ext-int. quasiconvex} \Rightarrow f \text{ ext-int. one convex}.$$

The characterization of all *ext-int. quasilinear* functions are obtained in Theorem 5.11. Note that it is also easy to see that these are precisely the 'Null Lagrangians' in this case. The theorem establishes the expected result that

$$f \text{ ext-int. polyaffine} \Leftrightarrow f \text{ ext-int. quasilinear} \Leftrightarrow f \text{ ext-int. one affine}.$$

It also says is that every *ext-int. quasilinear* (or *ext-int. polyaffine* or *ext-int. one affine*) functions are a sum of an *ext. quasilinear* (or *ext. polyaffine* or *ext. one affine*) function and an *int. quasilinear* (or *int. polyaffine* or *int. one affine*) function. This is striking since it means at the level of notions of affinity, no new nonlinear functionals pop up by considering both d and

δ together. This surprising situation is, in a sense, kind of like the situation for higher order derivatives in the classical case, considered in Ball-Currie-Olver [6], where no new quasilinear functions arise as well. Here this surprise is magnified by the fact that it also says either the ext. quasilinear part or the int. quasilinear part can be nonlinear, but not both. More precisely, if an ext-int. quasilinear function is a sum of a nonlinear ext. quasilinear function and an int. quasilinear function, then the int. quasilinear part is necessarily affine and vice versa. However, this is not the case at the level of notions of convexity. More precisely, though every ext-int. polyaffine function is the sum of an ext. polyaffine and an int. polyaffine function, an ext-int. polyconvex function need not be just a sum of an ext. polyconvex and an int. polyconvex one. For example, the function $f : \Lambda^4(\mathbb{R}^4) \times \Lambda^0(\mathbb{R}^4) \rightarrow \mathbb{R}$ given by ,

$$f(\xi, \eta) = \exp(|\xi \wedge \xi|^2 + \eta^2) \quad \text{for every } \xi \in \Lambda^4(\mathbb{R}^4), \eta \in \Lambda^0(\mathbb{R}^4),$$

is clearly not a sum of an ext. polyconvex function in the ‘first’ variable and an int. polyconvex function in the ‘second’ variable. But it is ext-int. polyconvex and hence ext-int. quasiconvex and ext-int. one convex as well, though not convex.

Existence theorems As functionals of the form (1.3) are coercive in $W^{1,p}$ with the usual growth assumption, as long as we impose the appropriate boundary conditions, the only issue we need to address is the weak lower semicontinuity on $W^{1,p}$. The functionals of the form

$$\int_{\Omega} f(x, \omega, d\omega, \delta\omega)$$

are weakly lower semicontinuous in $W^{1,p}$ if $(\xi, \eta) \mapsto f(x, \omega, \xi, \eta)$ is ext-int. quasiconvex for every $\omega \in \Lambda^k$ for a.e $x \in \Omega$, with usual growth assumptions. Note that unlike the case of only d or only δ , here explicit dependence on ω can be handled as long as it satisfies the usual growth restrictions. Theorem 5.21 and theorem 5.22 give the existence results.

1.8 Relationship with the classical calculus of variations

Ext. convexity notions as a special case of classical convexity notions The relationship of the convexity notions introduced here with the classical notions of rank one convexity, quasiconvexity and polyconvexity is an interesting one. We have seen already that the notions we introduced play analogous roles, but whether they are related to each other in any explicit sense is a reasonable question, which actually has a startlingly elegant answer. Before we can summarize the answers, we first need to analyze the specific algebraic structure of exterior forms in greater detail. To accomplish this, we introduce a projection mappings from $\Lambda^{k-1}(\mathbb{R}^n) \times \mathbb{R}^n$, which we identify with $\mathbb{R}^{\binom{n}{k-1} \times n}$, to $\Lambda^k(\mathbb{R}^n)$. The main idea behind introducing the map is that a $k-1$ -form is a map from $\Omega \subset \mathbb{R}^n$ to $\Lambda^{k-1}(\mathbb{R}^n)$, so it has $\mathbb{R}^{\binom{n}{k-1}}$ components. If we take the usual gradient of these components, the gradient takes values in $\mathbb{R}^{\binom{n}{k-1} \times n}$, whereas the exterior derivative of a $k-1$ -form is a k -form and hence takes values in $\Lambda^k(\mathbb{R}^n)$. Thus, to study the relationship it is useful to find a projection map from $\mathbb{R}^{\binom{n}{k-1} \times n}$ to $\Lambda^k(\mathbb{R}^n)$, which takes the gradient of a $k-1$ -form to its exterior derivative. We introduce such a projection map, denoted

by π^{ext} (or sometimes $\pi^{\text{ext},k}$ when we want to emphasize that the image is a k -form), where *ext* stands for the exterior derivative, such that

$$\pi^{\text{ext}}(\nabla\omega) = d\omega \quad \text{for every } \omega \in W^{1,1}(\Omega; \Lambda^{k-1})$$

When $k = 2$, this projection coincides with the standard alternating projection or skew-symmetric projection map which sends an $n \times n$ matrix A to its skew-symmetric part $\frac{1}{2}(A - A^T)$, where A^T is the transpose of A . The map π^{ext} actually also has the property that for any $a \in \mathbb{R}^n$ (seen both as a vector and a 1-form) and any $b \in \Lambda^{k-1}$, thought of also as a vector in $\mathbb{R}^{\binom{n}{k-1}}$, we have,

$$\pi^{\text{ext}}(a \otimes b) = a \wedge b.$$

These two properties immediately imply that for any map $f : \Lambda^k \rightarrow \mathbb{R}$, we have,

$$f \text{ ext. quasiconvex} \Leftrightarrow f \circ \pi^{\text{ext}} \text{ quasiconvex}$$

and

$$f \text{ ext. one convex} \Leftrightarrow f \circ \pi^{\text{ext}} \text{ rank one convex.}$$

This strongly hints that the statement

$$f \text{ ext. polyconvex} \Leftrightarrow f \circ \pi^{\text{ext}} \text{ polyconvex}$$

might also be true. Indeed it is true, but it is much harder to prove and is actually the nontrivial part of Theorem 3.54. The proof involves obtaining a formula for connecting wedge powers of $\pi^{\text{ext}}(X)$ with adjugates of the matrix $X \in \mathbb{R}^{\binom{n}{k-1} \times n}$ and a few algebraic niceties.

Similarly, we can define a projection map from $\mathbb{R}^{\binom{n}{k+1} \times n}$ to $\Lambda^k(\mathbb{R}^n)$, which takes the gradient of a $k+1$ -form to its interior derivative. Such a projection map, denoted by π^{int} (or sometimes $\pi^{\text{int},k}$ when we want to emphasize that the image is a k -form), where *int* stands for the interior derivative, has the property that

$$\pi^{\text{int}}(\nabla\omega) = \delta\omega \quad \text{for every } \omega \in W^{1,1}(\Omega, \Lambda^{k+1}).$$

Once again, the same map actually also has the property that for any $a \in \mathbb{R}^n$ (seen both as a vector and a 1-form) and any $b \in \Lambda^{k+1}$, thought of also as a vector in $\mathbb{R}^{\binom{n}{k+1}}$, we have,

$$\pi^{\text{int}}(a \otimes b) = a \lrcorner b.$$

These two properties immediately imply that for any map $f : \Lambda^k \rightarrow \mathbb{R}$, we have,

$$f \text{ int. quasiconvex} \Leftrightarrow f \circ \pi^{\text{int}} \text{ quasiconvex}$$

and

$$f \text{ int. one convex} \Leftrightarrow f \circ \pi^{\text{int}} \text{ rank one convex.}$$

We do not prove directly the result

$$f \text{ int. polyconvex} \Leftrightarrow f \circ \pi^{\text{int}} \text{ polyconvex,}$$

since all three statements actually follows from the corresponding ones for ext. quasiconvexity, ext. polyconvexity and ext. one convexity by Hodge duality.

Special structure of the ext. convexity notions So we see with this analysis how the results about functionals of the type $\int_{\Omega} f(d\omega)$ relate to the classical case $\int_{\Omega} f(\nabla\omega)$, where one views the $k-1$ -form ω as a vector-valued function taking values in $\mathbb{R}^{\binom{n}{k-1}}$. Since

$$\int_{\Omega} f(d\omega) = \int_{\Omega} (f \circ \pi^{\text{ext}})(\nabla\omega),$$

one might be tempted to view the theory for the functionals $\int_{\Omega} f(d\omega)$ as a corollary of the classical calculus of variations. However, this is not the case. The projection map has a very special algebraic structure which makes the ext. convexity notions very different from classical convexity notions. For example, if $F : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \mathbb{R}$ is polyconvex (respectively quasiconvex or rank one convex) and if there exists a map $f : \Lambda^k \rightarrow \mathbb{R}$ such that

$$F = f \circ \pi^{\text{ext},k},$$

then f is ext. polyconvex (respectively ext. quasiconvex or ext. one convex). But this last requirement is a very strong condition which forces such a function F to have additional properties, which are not at all typical for a general polyconvex (respectively quasiconvex or rank one convex) function. Any polyconvex (respectively quasiconvex or rank one convex) function $F : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \mathbb{R}$ need not be of the form $f \circ \pi^{\text{ext},k}$ with f ext. polyconvex (respectively ext. quasiconvex or ext. one convex). For example, for $n = k = 2$, the function

$$F(\Xi) = d \det \Xi \quad \text{for every } \Xi \in \mathbb{R}^{2 \times 2},$$

is polyconvex (and thus quasiconvex and rank one convex) for every $d \in \mathbb{R}$. If $d \neq 0$, there is however no function $f : \Lambda^2(\mathbb{R}^2) \rightarrow \mathbb{R}$ (in particular no ext. one convex and thus no ext. quasiconvex and no ext. polyconvex function f) such that $F = f \circ \pi^{\text{ext},2}$. Indeed if such an f exists, it can be shown that we must have $d = 0$. Many such manifestations of the special structure of the projection maps are also apparent in Theorem 3.37. The example given above is only a particular case of the fact that when $k = n$, there are no nonconvex quasiconvex function $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ that can be of the form $F = f \circ \pi^{\text{ext},n}$, for any function $f : \Lambda^n(\mathbb{R}^n) \rightarrow \mathbb{R}$. Similarly, if $k = n - 1$, there are no nonconvex quasiconvex function $F : \mathbb{R}^{\binom{n}{n-2} \times n} \rightarrow \mathbb{R}$ that

can be of the form $F = f \circ \pi^{\text{ext}, n-1}$, for any function $f : \Lambda^{n-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$. The case for $k = n - 2$ with n odd is similar. But in all those cases, if we do not require the restriction that it must be of the form $f \circ \pi^{\text{ext}, k}$, there exist many nonconvex quasiconvex functions.

None of the conclusions of Theorem 3.37 and Theorem 3.30, except Theorem 3.30(i), can be derived from the classical case using the equivalence theorem (Theorem 3.54). With hindsight, the structure theorem for ext. quasilinear functions (Theorem 3.20) can be deduced as a corollary of the classical result for quasilinear functions (Theorem 5.20 in [25]), but the proof given in this thesis is not only a direct one, but also considerably shorter. The only results we can obtain relatively cheaply from the classical results via the equivalence theorem (Theorem 3.54) are the semicontinuity results in section 3.6 (see for example, theorem 3.58), but these do not require the full conclusion of theorem 3.54 and can also be deduced independently as a special case of the semicontinuity results for vectorially ext. quasiconvex functions (see Theorem 4.25).

Ext-int. convexity notions and classical convexity notions We can also define a projection map which maps the gradient of a k -form to its exterior and interior derivative by taking both the exterior and interior projections together. We denote this projection map by $\pi^{\text{ext-int}, k}$ emphasizing that the argument is a k -form. The map $\pi^{\text{ext-int}, k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ has the property that,

$$\pi^{\text{ext-int}, k}(\nabla \omega) = \left(\pi^{\text{ext}, k+1}(\nabla \omega), \pi^{\text{int}, k-1}(\nabla \omega) \right) = (d\omega, \delta\omega) \quad \text{for every } \omega \in W^{1,1}(\Omega, \Lambda^k).$$

We also have, for any $a \in \mathbb{R}^n$ (seen both as a vector and a 1-form) and any $b \in \Lambda^k$, thought of also as a vector in $\mathbb{R}^{\binom{n}{k}}$,

$$\pi^{\text{ext-int}, k}(a \otimes b) = \left(\pi^{\text{ext}, k+1}(a \otimes b), \pi^{\text{int}, k-1}(a \otimes b) \right) = (a \wedge b, a \lrcorner b).$$

These two properties immediately imply that for any map $f : \Lambda^{k+1} \times \Lambda^{k-1} \rightarrow \mathbb{R}$, we have,

$$f \text{ ext-int. one convex} \Leftrightarrow f \circ \pi^{\text{ext-int}, k} \text{ rank one convex}$$

and

$$f \text{ ext-int. quasiconvex} \Leftrightarrow f \circ \pi^{\text{ext-int}, k} \text{ quasiconvex}.$$

These two results help us to derive the semicontinuity results for ext-int. quasiconvex functions from the classical results about semicontinuity of quasiconvex functions.

Classical convexity notions as a special case of vectorial ext. convexity notions The theory for classical calculus of variations for $\int_{\Omega} f(\nabla u)$, where u is a vector-valued function taking values in \mathbb{R}^N for some N , can be viewed as a special case of the functional $\int_{\Omega} f(d\omega_1, \dots, d\omega_m)$. We just view each component of u , which are real-valued functions, as 0-forms. This connection

is made explicit in proposition 4.11. For any integer $m \geq 1$, by seeing $\xi_i \in \Lambda^1$ as a vector in \mathbb{R}^n , which in turn is viewed as the i -th row of an $m \times n$ matrix Ξ , and conversely, by viewing each row of an $m \times n$ matrix Ξ as a 1-form, any function

$$f : \underbrace{\Lambda^1 \times \dots \times \Lambda^1}_{m \text{ times}} \rightarrow \mathbb{R},$$

given by,

$$(\xi_1, \dots, \xi_m) \mapsto f(\xi_1, \dots, \xi_m) \quad \text{for every } (\xi_1, \dots, \xi_m) \in \underbrace{\Lambda^1 \times \dots \times \Lambda^1}_{m \text{ times}},$$

can also be viewed as the map

$$\Xi \mapsto f(\Xi) \quad \text{for every } \Xi \in \mathbb{R}^{m \times n}.$$

The proposition says that under this identification, we have,

$$\begin{aligned} f : \underbrace{\Lambda^1 \times \dots \times \Lambda^1}_{m \text{ times}} \rightarrow \mathbb{R} \text{ is vectorially ext. polyconvex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is polyconvex,} \\ f : \underbrace{\Lambda^1 \times \dots \times \Lambda^1}_{m \text{ times}} \rightarrow \mathbb{R} \text{ is vectorially ext. quasiconvex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is quasiconvex,} \\ f : \underbrace{\Lambda^1 \times \dots \times \Lambda^1}_{m \text{ times}} \rightarrow \mathbb{R} \text{ is vectorially ext. one convex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is rank one convex.} \end{aligned}$$

Thus, the structure theorem for vectorial ext. quasilinear functions (Theorem 4.12) immediately imply, in particular, the classical result for quasilinear functions (cf. theorem 5.20 in [25]).

The semicontinuity results in the classical case, for example theorem 8.4 in [25], which has been established by Morrey [52], [53] under additional hypotheses and has been refined by Meyers [49], Acerbi-Fusco [1] and Marcellini [48], follows as a particular case of theorem 4.22. However, also theorem 8.8 and theorem 8.11 in [25] can be derived from the a semicontinuity result which we state in theorem 4.27.

1.9 Maxwell operator

In the second part of this thesis, we study a number of boundary value problems for partial differential equations for differential forms. Since a differential form always has several components unless it is a 0-form, these ‘equations’ are actually systems of partial differential equations.

A simple example of the type of systems we shall be studying is,

$$\delta(A(x)(d\omega)) = f \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is an open, smooth, bounded set and A is a matrix field on Ω . This is of course a linear system of second order partial differential equations. The system (1.4) shall be called

linear Maxwell equation for k -forms in this thesis. The name derives from the fact when $k = 1$, $n = 3$ and $A(x) \equiv \mathbf{I}$, the system becomes, up to a sign,

$$\operatorname{curl} \operatorname{curl} E = f,$$

for an unknown vector field E . This important equation in physics is called the *time-harmonic Maxwell's equation*. In fact, lots of essential features of the general system (1.4) are already present at the level of 1-form. This, however is not true for the case of 0-forms. When $k = 0$, the system (1.4) reduces to, the equation

$$\operatorname{div}(A(x)(\nabla u)) = f,$$

for an unknown function u . Though this equation is the central object of study in the theory of linear elliptic partial differential equations, it is considerably easier to handle than (1.4).

We shall also be interested in semilinear or quasilinear versions of the Maxwell's equation for k -forms. In particular, a system of the form

$$\delta(A(x)(d\omega)) = f(\omega) \quad \text{in } \Omega, \quad (1.5)$$

is called *semilinear Maxwell equation for k -forms*. At the level of 0-forms, the well-studied semilinear Poisson problem

$$-\Delta u = |u|^{p-2}u \quad \text{in } \Omega$$

is the prototype equation for most of the theory of semilinear elliptic equations. Following the same practice, we shall mostly be interested in power-type nonlinearity, i.e the cases when $f(\omega) = \pm|\omega|^{p-2}\omega$.

Likewise, a system of the form

$$\delta(A(x, d\omega)) = f \quad \text{in } \Omega, \quad (1.6)$$

is called *quasilinear Maxwell equation for k -forms*. A particularly important example is the case when $A(x, d\omega) = |d\omega|^{p-2}d\omega$ and $f = 0$, when the equation reduces to

$$\delta(|d\omega|^{p-2}d\omega) = 0 \quad \text{in } \Omega,$$

whose solutions are called *p -harmonic fields*. Also, at the level of 0-forms, both the p -Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

and the ϱ -harmonic equation

$$\operatorname{div}(\varrho(\nabla u)) = 0$$

are particular cases of equation (1.6). Although the techniques involved are basically variational in nature in all cases, sometimes our hypotheses will allow treatment of cases which need not

come from a minimization problem.

Full Dirichlet boundary data for linear and quasilinear case We solve the full Dirichlet data boundary value problem for the linear and quasilinear Maxwell operator. More precisely, the boundary value problem in the linear case is,

$$\begin{cases} \delta(A(x)(d\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega, \end{cases} \quad (1.7)$$

where $A(x)$ is a matrix field and for the quasilinear case is

$$\begin{cases} \delta(A(x, d\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega, \end{cases} \quad (1.8)$$

where A is a nonlinear with respect to the second variable. The difficulty in both cases is that the operators, with usual hypothesis on A are not elliptic. They have a large infinite dimensional kernel, as any closed differential form with zero boundary values is in the kernel. But we shall see that this freedom is essentially what allows us to solve the full Dirichlet data problem. However both results can be proved by minimization techniques with appropriate assumptions on A . To weaken the hypothesis on A somewhat, we prove these two results directly. In the linear case, we use a decomposition result coupled with Lax-Milgram theorem to show the existence of the elliptic boundary value problem

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.9)$$

We then use this existence result to solve the full Dirichlet data problem (1.7). Though for that we only need the case $\lambda = 0$, we however show the existence for the general system (1.9).

Using the same decomposition result and the monotone operator theory, we show the existence for the quasilinear system

$$\begin{cases} \delta(A(x, d\omega)) = f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.10)$$

This system is important in its own right. A special case of this system for k -forms, which we obtain by taking $A(x, \xi) = \rho(|\xi|^2)\xi$ for some function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ for every $\xi \in \Lambda^{k+1}$ and $f = 0$, is the system,

$$\begin{cases} \delta(\rho(|d\omega|^2)d\omega) = 0 \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.11)$$

For every solution ω of this system, its exterior derivative $v = d\omega$ satisfies,

$$\begin{cases} \delta(\varrho(|v|^2)v) = 0 & \text{and} & dv = 0 \text{ in } \Omega, \\ \nu \wedge v = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.12)$$

Such forms v that solve (1.12) are called *ϱ -harmonic Dirichlet k -forms*. Conversely, when Ω is a contractible domain, every solution v of (1.12) can be written as $v = d\omega$, where ω solves (1.11). Thus, under this identification the two systems are equivalent on contractible domains. Along with ϱ -harmonic Dirichlet and Neumann k -forms, ϱ -harmonic k -forms, has been studied before by a number of authors, most notably in the celebrated paper by Uhlenbeck [70] and also by Hamburger [35] (see also Beck-Stroffolini [14]).

We use existence result for (1.10) to prove the existence for the full Dirichlet boundary value problem (1.8). Exactly the similar analysis applies to the dual problems

$$\begin{cases} d(A(x)(\delta\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d(A(x)(\delta\omega)) = \lambda\omega + f \text{ in } \Omega, \\ d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega, \end{cases}$$

and of course also to

$$\begin{cases} d(A(x, \delta\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d(A(x, \delta\omega)) = f \text{ in } \Omega, \\ d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Regularity results for linear system and consequences We also study the up to the boundary $W^{r,2}$ regularity results for the system

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega, \end{cases}$$

which also applies to the system

$$\begin{cases} \delta(A(x)(d\omega)) + \delta d\omega = \lambda\omega + f & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega, \\ \nu \wedge \delta\omega = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.13)$$

This last system can be viewed as a generalization of the Hodge Laplacian system

$$\begin{cases} \Delta\omega = \lambda\omega + f & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega, \\ \nu \wedge \delta\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta = \delta d + d\delta$ here denotes the Hodge Laplacian. Of course, interior regularity results, both in L^p and Hölder space settings follow quite easily from the corresponding ones for linear elliptic system, but there is no easy way to obtain the up to the boundary results from the usual theory of linear elliptic systems, because of the special nature of the boundary conditions. Up to the boundary $W^{r,2}$ regularity results for the system (1.13), as far as we are aware, are new.

This analysis also allows us to solve, in $W^{r,2}$ spaces the following first order systems

$$\begin{cases} d(A(x)\omega) = f & \text{and } \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \wedge A(x)\omega = \nu \wedge \omega_0 & & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d(A(x)\omega) = f & \text{and } \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \lrcorner B(x)\omega = \nu \lrcorner \omega_0 & & \text{on } \partial\Omega. \end{cases}$$

Of course, when both $A(x) = B(x) \equiv \mathbf{I}$, the system reduces to

$$\begin{cases} d\omega = f & \text{and } \delta\omega = g & \text{in } \Omega, \\ \text{with either} & & \\ \nu \wedge \omega = \nu \wedge \omega_0 & & \text{on } \partial\Omega, \\ \text{or} & & \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 & & \text{on } \partial\Omega. \end{cases}$$

These systems are called div-curl systems or sometimes the Cauchy-Riemann systems. In this special case however, regularity results up to the boundary can be proved in $W^{r,p}$ and $C^{r,\alpha}$ also. These results follow from the Hodge-Morrey decomposition, a consequence of the regularity results of the Hodge-Laplacian, originally due to Morrey (cf. [53]). The derivation of $W^{r,p}$ and $C^{r,\alpha}$ regularity results for these systems from the Hodge-Morrey decomposition is well-known (cf. Csató-Dacorogna-Kneuss [21] for the results except $W^{r,p}$ regularity for $1 < p < 2$ and cf. Subsection 2.5.2 of this thesis, for this case).

Eigenvalue problem for the Semilinear Maxwell operator We also study the eigenvalue problem for semilinear Maxwell operator, i.e the boundary value problem

$$\begin{cases} \delta(A(x)(d\omega)) + |\omega|^{p-2}\omega = \lambda\omega \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.14)$$

However, if the semilinearity had a different sign, as in the boundary value problem

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + |\omega|^{p-2}\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega, \end{cases}$$

then the sign of the semilinearity makes the lower order term in the energy functional convex, coercive and nonnegative. Hence, direct minimization techniques apply and we can show existence of a solution for any f , any boundary value ω_0 , but only for nonnegative λ away from the spectrum of the principal part of the operator, which is linear. Though we show existence by using monotone operator theory, to weaken the hypotheses a bit.

But the original eigenvalue problem (1.14) is much harder. In fact, this problem we are only able to solve for a range of λ , for λ in the real half-line containing the spectrum of the principal linear part. The techniques are also completely different. Here the energy functional is unbounded both above and below and hence minimization techniques do not apply. We use critical point techniques to show that the energy functional admits a saddle-type critical point. However, in contrast to the case $k = 0$, when the equation is

$$-\Delta u = \lambda u + |u|^{p-2}u,$$

the energy functional for (1.14) is indefinite on an infinite dimensional subspace as soon as $1 \leq k \leq n - 1$. So usual critical point theory also does not apply. We use the technique of ‘Nehari-Pankov’ manifolds, suitably modified. When $k = 1, n = 3$ and $A(x) \equiv \mathbf{I}$, (1.14) reduces to

$$\begin{cases} \operatorname{curl} \operatorname{curl} \omega + |\omega|^{p-2}\omega = \lambda\omega \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega, \end{cases}$$

The existence result in this case, using the technique of ‘Nehari-Pankov’ manifolds, is obtained in Bartsch-Mederski [13]. We generalize their results to handle the more general case. Also, even when $k = 1$ and $n = 3$, the result presented here is new and slightly more general than [13], as it can handle systems of the form

$$\operatorname{curl}(A(x) \operatorname{curl} \omega) + |\omega|^{p-2}\omega = \lambda\omega,$$

when $A(x)$ need not even have constant coefficients.

1.10 Organization

We conclude this introduction with some remarks about the organization of the rest of the thesis, materials and notations. More often than not, the burden of notations is quite heavy while working with differential forms. So it is crucial to use good notations and shorthands to keep the calculations manageable and readable. Appendix A contains the notations used in this thesis.

Appendix B contains usual facts about the function spaces used. Chapter 2 contains the necessary background material, both algebraic and analytic, that is used in this thesis. Although most results are known and are stated without proof with a reference to articles or books where the proof can be found, there are some new results and full proofs are given for them.

Primary material is divided into two parts. Chapters 3, 4 and 5 constitute the first part, titled *Direct Methods in Calculus of Variations for Differential Forms*. Chapter 3 contains the analysis for functionals of the form

$$\int_{\Omega} f(d\omega),$$

including its relationship with the classical case of $\int_{\Omega} f(\nabla\omega)$ via the projection maps π^{ext} . Chapter 4 deals with the case for the functional

$$\int_{\Omega} f(d\omega_1, \dots, d\omega_m),$$

along with weak lower semicontinuity and weak continuity results. The case for functionals of the form

$$\int_{\Omega} f(d\omega, \delta\omega)$$

is contained in chapter 5.

Chapters 6 and 7 constitute the second part of this thesis, titled *Some Boundary value problems for Differential Forms*. Chapter 6 presents the existence and regularity results about linear Maxwell operator and the related boundary value problems for first and second order systems. Chapter 7 presents the existence results for the nonlinear Maxwell operator, starting with the semilinear operator, treats the different sign of the semilinearity separately and then presents the existence results for the quasilinear Maxwell operator.

Most of the results in the first part also appeared elsewhere, divided between the articles Bandyopadhyay-Dacorogna-Sil [10], Bandyopadhyay-Sil [11] and Bandyopadhyay-Sil [12].

Chapter 2

Differential forms

2.1 Introduction

The present chapter serves as the concise conglomeration of background material for the rest of this thesis. We start by describing the algebraic preliminaries of exterior forms and introduce a suitable notion of ‘differential forms’. For the rest of this thesis, we shall be using the term ‘differential forms’ to mean ‘differential forms with measurable components’, deviating from the common practice of using the term to mean ‘smooth differential forms’. We then introduce the function spaces which we shall use throughout our analysis of problems involving differential forms. We also record an extremely important inequality, called the Gaffney inequality and several important facts about these spaces. We shall, for the most part, restrict our attention to the cases where the domain is a bounded open subset of \mathbb{R}^n with smooth enough boundary, though most of the results stated in this section can be proved for a compact Riemannian manifold with boundary. Most of the material in this chapter is well known, though not always easy to find in one place in the available literature. We present the definitions and statements of the results and refer to the bibliography for proofs of well known results. Only lesser known or new results are proved in complete details.

The rest of the chapter is organized as follows. In Section 2.2, we define the exterior forms and the basic operations on them, namely the exterior product, interior product and Hodge star operator. We present the basic properties of these operations and finally we present a few results about the divisibility in the space of exterior forms. In section 2.3, we define the notion of differential forms that we shall use, namely differential forms with measurable components and define the exterior derivative and the codifferential. Section 2.4 discusses the function spaces of differential forms that we shall use. We define the partly Sobolev classes which are crucial for working with differential forms and summarize some of their properties. We also provide a definition for the Trace operator in these spaces and present the important Gaffney inequality. In the final section, i.e section 2.4.3, we present the Hodge-Morrey decompositions and derive some of its corollaries that will be useful later on.

2.2 Exterior forms

2.2.1 Definitions and main properties

Definition 2.1 (Exterior form) Let $k \geq 1$ be an integer. An exterior k -form over \mathbb{R}^n is an alternating k -linear map from \mathbb{R}^n to \mathbb{R} . More precisely, an exterior k -form ξ is a map

$$\xi : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} \rightarrow \mathbb{R}$$

such that,

1. ξ is linear in each variable

and

2. for every $X_1, \dots, X_k \in \mathbb{R}^n$ and for every permutation $\sigma \in S_k$, we have,

$$\xi(X_{\sigma(1)}, \dots, X_{\sigma(k)}) = \text{sgn}(\sigma) \xi(X_1, \dots, X_k).$$

We write $\Lambda^k(\mathbb{R}^n)$ to denote the set of exterior k -forms over \mathbb{R}^n . If $k = 0$, we set $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$. If we choose $\{e_1, \dots, e_n\}$ as a basis for \mathbb{R}^n , then we write its dual basis as $\{e^1, \dots, e^n\}$, which is a basis for $\Lambda^1(\mathbb{R}^n)$.

Definition 2.2 (Exterior product) Let $f \in \Lambda^k(\mathbb{R}^n)$ and $g \in \Lambda^l(\mathbb{R}^n)$. The exterior product of f and g , written as $f \wedge g$ is an exterior $(k+l)$ -form and is defined by,

$$(f \wedge g)(X_1, \dots, X_{k+l}) = \sum_{\sigma \in S_{k,l}} \text{sgn}(\sigma) f(X_{\sigma(1)}, \dots, X_{\sigma(k)}) g(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})$$

for every $X_1, \dots, X_{k+l} \in \mathbb{R}^n$ and for every permutation $\sigma \in S_{k,l}$, where $S_{k,l}$ is the subset of permutations defined by,

$$S_{k,l} := \{\sigma \in S_{k+l} : \sigma(1) < \dots < \sigma(k); \sigma(k+1) < \dots < \sigma(k+l)\}.$$

Note that If we choose $\{e_1, \dots, e_n\}$ as a basis for \mathbb{R}^n , $\{e^{i_1} \wedge \dots \wedge e^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$ is a basis for $\Lambda^k(\mathbb{R}^n)$ and this immediately yields, $\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}$.

From here onwards, the notation ξ^m , where ξ is a k -form and m is a positive integer will be employed to denote the exterior power of a form. More precisely,

$$\xi^m := \underbrace{\xi \wedge \dots \wedge \xi}_{m \text{ times}}.$$

We now list a few elementary properties of the exterior product.

Proposition 2.3 The exterior product is bilinear, associative and graded commutative. More precisely, if $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^l(\mathbb{R}^n)$ and $h \in \Lambda^p(\mathbb{R}^n)$ and $\lambda, \mu \in \mathbb{R}$, then we have the following:

- Bilinearity:

$$\begin{aligned}(\lambda f + \mu g) \wedge h &= \lambda f \wedge h + \mu g \wedge h, \\ f \wedge (\lambda g + \mu h) &= \lambda f \wedge g + \mu f \wedge h.\end{aligned}$$

- Associativity:

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

- Graded Commutativity:

$$f \wedge g = (-1)^{kl} g \wedge f.$$

Definition 2.4 (Hodge duality) *The Hodge star operator is the linear map*

$$* : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{n-k}(\mathbb{R}^n)$$

defined by

$$f \wedge g = \langle *f; g \rangle e^1 \wedge \dots \wedge e^n$$

for every $g \in \Lambda^{n-k}(\mathbb{R}^n)$.

The following properties are easy to verify.

Proposition 2.5 *Let $0 \leq k \leq n$ be an integer. Then for any $f, g \in \Lambda^k(\mathbb{R}^n)$ and for any $I \in \mathcal{T}^k$, $J \in \mathcal{T}^{n-k}$ such that $e^I \wedge e^J = (-1)^r e^1 \wedge \dots \wedge e^n$, we have,*

1. $*(e^I) = (-1)^r e^J$.
2. $*1 = e^1 \wedge \dots \wedge e^n$.
3. $*(e^1 \wedge \dots \wedge e^n) = 1$.
4. $*(f) = (-1)^{k(n-k)} f$.
5. $f \wedge (*g) = \langle f; g \rangle e^1 \wedge \dots \wedge e^n$.

Definition 2.6 (Interior product) *Let $0 \leq l \leq k \leq n$ be integers and $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^l(\mathbb{R}^n)$. The interior product $g \lrcorner f$ is a $(k-l)$ -form defined by,*

$$g \lrcorner f = (-1)^{n(k-l)} * (g \wedge (*f)).$$

We now record the following useful properties. For the proof, see Proposition 2.16 in [21].

Proposition 2.7 *Let $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^l(\mathbb{R}^n)$ and $h \in \Lambda^p(\mathbb{R}^n)$ with integers $0 \leq k, l, p \leq n$. Then*

$$(h \wedge g) \lrcorner f = (-1)^{k+l} h \lrcorner (g \lrcorner f).$$

Note that when $l + p > k$, the above identity holds trivially with both sides equal to zero. Furthermore, if $p = k + l$, then

$$\langle f \wedge g; h \rangle = (-1)^{l(k+1)} \langle g; f \lrcorner h \rangle = (-1)^k \langle f; g \lrcorner h \rangle.$$

If $\xi \in \Lambda^1(\mathbb{R}^n)$, then

$$\begin{aligned} \xi \lrcorner (f \wedge g) &= (\xi \lrcorner f) \wedge g + (-1)^{kl} (\xi \lrcorner g) \wedge f \\ &= (\xi \lrcorner f) \wedge g + (-1)^k f \wedge (\xi \lrcorner g). \end{aligned}$$

Again, if $\xi, \eta \in \Lambda^1(\mathbb{R}^n)$, then

$$\xi \lrcorner (\eta \wedge f) + \eta \wedge (\xi \lrcorner f) = \langle \xi; \eta \rangle f$$

and

$$\xi \lrcorner (\xi \wedge f) + \xi \wedge (\xi \lrcorner f) = |\xi|^2 f$$

and

$$|\xi|^4 |f|^2 = |\xi \lrcorner (\xi \wedge f)|^2 + |\xi \wedge (\xi \lrcorner f)|^2 = |\xi|^2 (|\xi \wedge f|^2 + |\xi \lrcorner f|^2).$$

2.2.2 Divisibility

Definition 2.8 Let $1 \leq k \leq n$ and $\xi \in \Lambda^k(\mathbb{R}^n)$.

(i) We say that ξ is 1-divisible if there exist $a \in \Lambda^1(\mathbb{R}^n)$ and $b \in \Lambda^{k-1}(\mathbb{R}^n)$ such that

$$\xi = a \wedge b.$$

(ii) We say that ξ is totally divisible if there exist $a_1, \dots, a_k \in \Lambda^1(\mathbb{R}^n)$ such that

$$\xi = a_1 \wedge \dots \wedge a_k.$$

Definition 2.9 Let $1 \leq k \leq n$ and $\xi \in \Lambda^k(\mathbb{R}^n)$. Let

$$\Lambda_\xi^1 := \left\{ u \in \Lambda^1 : \exists g \in \Lambda^{k-1} \text{ with } g \lrcorner \xi = u \right\}.$$

Then we define the rank of order 1 of ξ as,

$$\text{rank}_1[\xi] = \dim(\Lambda_\xi^1).$$

Now we present a few algebraic facts related to 1-divisibility. For the proofs, see Proposition 2.37 and Proposition 2.43 of [21].

Proposition 2.10 Let $1 \leq k \leq n$ and $\xi \in \Lambda^k(\mathbb{R}^n)$ with $\xi \neq 0$.

(i) Let $a \in \Lambda^1, a \neq 0$ be such that

$$a \wedge \xi = 0,$$

Then, ξ is 1-divisible, there exists a form $b \in \Lambda^{k-1}$ such that $\xi = a \wedge b$ and $a \in \Lambda_\xi^1$.

(ii) ξ is totally divisible if and only if

$$\text{rank}_1[\xi] = k$$

if and only if

$$b \wedge \xi = 0 \text{ for all } b \in \Lambda_\xi^1.$$

(iii) If $k = 2$, then $\text{rank}_1[\xi]$ is even and any even integer greater than or equal to k and less than or equal to n can be achieved. Moreover, $\text{rank}_1[\xi] = 2m$ if and only if

$$\xi^m \neq 0 \text{ and } \xi^{m+1} = 0.$$

(iv) If $3 \leq k \leq n$, then

$$\text{rank}_1[\xi] \in \{k, k+2, \dots, n\}$$

and any of the values can be achieved.

(v) $\text{rank}_1[\xi]$ can never be $k+1$. In particular, when $k = n-1$ then

$$\text{rank}_1[\xi] = n-1.$$

(vi) If k is odd and if $\text{rank}_1[\xi] = k+2$, then ξ is 1-divisible.

Remark 2.11 Note that (i) and (ii) implies that every $\xi \in \Lambda^n$ is 1-divisible and totally divisible. Also, (ii) and (v) together implies that every $\xi \in \Lambda^{n-1}$ is 1-divisible. Likewise (ii) and (vi) implies that if k is odd then every $\xi \in \Lambda^{n-2}$ is 1-divisible. Of course, every $\xi \in \Lambda^1$ is trivially 1-divisible.

2.3 Differential forms and their derivatives

Usually, differential forms are either defined or tacitly understood as meaning smooth differential forms, i.e smooth functions $\omega : \Omega \rightarrow \Lambda^k$. However, in this thesis, we are going to work with their nonsmooth cousins rather heavily.

Definition 2.12 (Differential form) Let $0 \leq k \leq n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. A differential k -form ω is a measurable function $\omega : \Omega \rightarrow \Lambda^k$. We write $\omega \in \mathcal{M}(\Omega; \Lambda^k)$.

We now define two important operations on differential forms which form the basis of exterior differential calculus. We start with the definition of exterior derivative. Our definition is very similar to the usual definition of weak derivative.

Definition 2.13 (Exterior derivative) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $\omega \in L_{\text{loc}}^1(\Omega; \Lambda^k)$. A differential $(k+1)$ -form $\varphi \in L_{\text{loc}}^1(\Omega; \Lambda^{k+1})$ is called the

exterior derivative of ω , denoted by $d\omega$, if

$$\int_{\Omega} \eta \wedge \varphi = (-1)^{n-k} \int_{\Omega} d\eta \wedge \omega,$$

for all $\eta \in C_0^\infty(\Omega; \Lambda^{n-k-1})$.

Remark 2.14 Let $\omega = \sum_{I \in \mathcal{T}_k} a_I dx_I \in W_{\text{loc}}^{1,p}(\Omega; \Lambda^k)$. Then, for every $I = (i_1, \dots, i_{k+1}) \in \mathcal{T}_{k+1}$, we have that

$$(d\omega)_{(i_1, \dots, i_{k+1})} = \sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \frac{\partial a_{i_1, \dots, i_{\gamma-1}, i_{\gamma+1}, \dots, i_{k+1}}}{\partial x_{i_\gamma}}, \text{ a.e. on } \Omega,$$

where $\frac{\partial a_{i_1, \dots, i_{\gamma-1}, i_{\gamma+1}, \dots, i_{k+1}}}{\partial x_{i_\gamma}}$ are weak derivatives of $a_{i_1, \dots, i_{\gamma-1}, i_{\gamma+1}, \dots, i_{k+1}}$.

The formal adjoint of d gives us another extremely important operator to look at.

Definition 2.15 (Hodge codifferential) Let $1 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be open and let $\omega \in L_{\text{loc}}^1(\Omega; \Lambda^k)$ be such that $d\omega$ exists. Then, the Hodge codifferential of ω is a $(k-1)$ -form $\delta\omega \in L_{\text{loc}}^1(\Omega; \Lambda^{k-1})$ defined as

$$\delta\omega := (-1)^{nk+1} * d * \omega.$$

Remark 2.16 Let $\omega = \sum_{I \in \mathcal{T}_k} a_I dx_I \in W_{\text{loc}}^{1,p}(\Omega; \Lambda^k)$. Then, for every $I = (i_1, \dots, i_{k-1}) \in \mathcal{T}_{k-1}$, we have that

$$(\delta\omega)_{(i_1, \dots, i_{k-1})} = \sum_{\gamma=1}^k (-1)^{\gamma-1} \sum_{i_{\gamma-1} < j < i_\gamma} \frac{\partial a_{i_1, \dots, i_{\gamma-1}, j, i_{\gamma+1}, \dots, i_k}}{\partial x_j}, \text{ a.e. on } \Omega,$$

where $\frac{\partial a_{i_1, \dots, i_{\gamma-1}, j, i_{\gamma+1}, \dots, i_k}}{\partial x_j}$ are understood as weak derivatives.

Remark 2.17 Of course, for smooth differential forms, both these operations coincide with the usual exterior derivative and the codifferential.

2.4 Function Spaces of differential forms on \mathbb{R}^n

Various function spaces of differential forms like the Lebesgue spaces $L^p(\Omega, \Lambda^k)$, Sobolev spaces $W^{r,p}(\Omega; \Lambda^k)$, Hölder spaces $C^{r,\alpha}(\Omega; \Lambda^k)$ etc are defined in the usual way with the obvious norms by requiring each component to lie in the scalar versions of the corresponding spaces. For the sake of completeness, we briefly recall their definitions and state a few useful properties of these spaces in Appendix B.

2.4.1 Partly Sobolev classes

In addition to the usual Sobolev spaces $W^{m,p}(\Omega; \Lambda^k)$, there are some additional Sobolev type spaces specifically suitable for forms. The reason for introducing these spaces springs from the observation that the partial differentiation on forms occurs only via operators d and δ . We first introduce partial Sobolev spaces of first order. See [40] for more detail.

Definition 2.18 (Partial Sobolev spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We define $W^{d,p}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$ and $d\omega \in L^p(\Omega; \Lambda^{k+1})$. It is endowed with the norm

$$\|\omega\|_{d,p} := \|\omega\|_p + \|d\omega\|_p, \text{ for all } \omega \in W^{d,p}(\Omega; \Lambda^k).$$

Similarly, for $1 \leq k \leq n$, we define $W^{\delta,p}(\Omega; \Lambda^k)$ as the space of differential k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\delta,p} := \|\omega\|_p + \|\delta\omega\|_p, \text{ for all } \omega \in W^{\delta,p}(\Omega; \Lambda^k).$$

It is often useful in nonlinear problems to introduce another type of partial Sobolev spaces.

Definition 2.19 (Partial Sobolev spaces of (p, q) type) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p, q \leq \infty$. We define $W^{d,p,q}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^q(\Omega; \Lambda^k)$ and $d\omega \in L^p(\Omega; \Lambda^{k+1})$, endowed with the norm

$$\|\omega\|_{d,p,q} := \|\omega\|_q + \|d\omega\|_p, \text{ for all } \omega \in W^{d,p,q}(\Omega; \Lambda^k).$$

Similarly, for $1 \leq k \leq n$, we define $W^{\delta,p,q}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^q(\Omega; \Lambda^k)$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\delta,p,q} := \|\omega\|_q + \|\delta\omega\|_p, \text{ for all } \omega \in W^{\delta,p,q}(\Omega; \Lambda^k).$$

Remark 2.20 Of course, when $p = q$, we have $W^{d,p,p}(\Omega; \Lambda^k) = W^{d,p}(\Omega; \Lambda^k)$, $W^{\delta,p,p}(\Omega; \Lambda^k) = W^{\delta,p}(\Omega; \Lambda^k)$.

There is also another class of Sobolev spaces involving both operators d and δ .

Definition 2.21 (Total Sobolev spaces) Let $1 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We define $\mathcal{L}^{1,p}(\Omega; \Lambda^k)$ to be the space of k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$, $d\omega \in L^p(\Omega; \Lambda^{k+1})$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\mathcal{L}^{1,p}} := \|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p, \text{ for all } \omega \in \mathcal{L}^{1,p}(\Omega; \Lambda^k).$$

2.4.2 Trace on partial Sobolev spaces

The notion of trace on partial Sobolev spaces play an important role in the subsequent discussion. We begin with the following definitions.

Definition 2.22 Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $1 \leq p < \infty$. We define

$$\begin{aligned} W_T^{d,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{d,p}(\Omega; \Lambda^k) : \int_{\Omega} \langle d\omega; \phi \rangle = - \int_{\Omega} \langle \omega; \delta\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k+1}) \right\}. \\ W_N^{\delta,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{\delta,p}(\Omega; \Lambda^k) : \int_{\Omega} \langle \delta\omega; \phi \rangle = - \int_{\Omega} \langle \omega; d\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k-1}) \right\}. \end{aligned}$$

Remark 2.23 Note that, if $\omega \in W_T^{d,p}(\Omega; \Lambda^k)$, we have that $d\omega \in W_T^{d,p}(\Omega; \Lambda^{k+1})$. Similar statement holds in $W_N^{\delta,p}(\Omega; \Lambda^k)$.

Definition 2.24 We set

$$W_T^{d,\infty}(\Omega; \Lambda^k) = W^{d,\infty}(\Omega; \Lambda^k) \cap W_T^{d,1}(\Omega; \Lambda^k).$$

Definition 2.25 Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $1 \leq p, q < \infty$. We define,

$$\begin{aligned} W_T^{d,p,q}(\Omega; \Lambda^k) &= \left\{ \omega \in W^{d,p,q}(\Omega; \Lambda^k) : \int_{\Omega} \langle d\omega; \phi \rangle = - \int_{\Omega} \langle \omega; \delta\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k+1}) \right\}. \\ W_N^{\delta,p,q}(\Omega; \Lambda^k) &= \left\{ \omega \in W^{\delta,p}(\Omega; \Lambda^k) : \int_{\Omega} \langle \delta\omega; \phi \rangle = - \int_{\Omega} \langle \omega; d\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k-1}) \right\}. \end{aligned}$$

Now we define the trace maps. The following theorems were proved in [50]. See Proposition 4.1 in [50].

Theorem 2.26 (Tangential trace in $W^{d,p}(\Omega; \Lambda^k)$) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and let $1 < p < \infty$. The map $\text{Tr}_T : W^{d,p}(\Omega; \Lambda^k) \rightarrow W^{-\frac{1}{p},p}(\partial\Omega; \Lambda^{k+1})$ defined via the duality pairing

$$\int_{\partial\Omega} \langle \text{Tr}_T(\omega); \text{Tr}(\phi) \rangle = \int_{\Omega} \langle d\omega; \phi \rangle + \int_{\Omega} \langle \omega; \delta\phi \rangle,$$

for all $\omega \in W^{d,p}(\Omega; \Lambda^k)$, $\phi \in W^{1,p'}(\Omega; \Lambda^{k+1})$, is a well-defined, bounded linear operator.

Theorem 2.27 (Normal trace in $W^{\delta,p}(\Omega; \Lambda^k)$) Let $1 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and let $1 < p < \infty$. The map $\text{Tr}_N : W^{\delta,p}(\Omega; \Lambda^k) \rightarrow W^{-\frac{1}{p},p}(\partial\Omega; \Lambda^{k-1})$ defined via the duality pairing

$$\int_{\partial\Omega} \langle \text{Tr}_N(\omega); \text{Tr}(\phi) \rangle = \int_{\Omega} \langle \delta\omega; \phi \rangle + \int_{\Omega} \langle \omega; d\phi \rangle,$$

for all $\omega \in W^{\delta,p}(\Omega; \Lambda^k)$, $\phi \in W^{1,p'}(\Omega; \Lambda^{k-1})$, is a well-defined, bounded linear operator.

Remark 2.28 1. In Theorems 2.26 and 2.27, p' is the Hölder conjugate exponent of p and $\text{Tr} : W^{1,p'}(\Omega; \Lambda^{k\pm 1}) \rightarrow W^{\frac{1}{p},p'}(\partial\Omega; \Lambda^{k\pm 1})$ is the usual Sobolev trace map.

2. See [50] for a precise description of the images of the maps Tr_T and Tr_N . A particularly important detail concerning this is unlike the usual trace map, the tangential and normal trace maps are not onto, in general. We would not be encountering this fact anymore, but it is important to point out that this is a chief reason why in all the theorems for boundary value problems appearing later in this thesis, it will be explicitly assumed that the given boundary value actually is the trace of a given differential form.

Theorems 2.26 and 2.27 lead us to the definition of the tangential and normal components.

Definition 2.29 (Tangential and normal components) Let $0 \leq k \leq n$, let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain.

1. If $\omega \in W^{d,p}(\Omega; \Lambda^k)$, we say that $\text{Tr}_T(\omega)$ is the tangential component of ω on $\partial\Omega$.
2. If $\omega \in W^{\delta,p}(\Omega; \Lambda^k)$, we say that $\text{Tr}_N(\omega)$ is the normal component of ω on $\partial\Omega$.

Remark 2.30 Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and let $1 < p < \infty$. Note that,

1. If $\omega \in W^{1,p}(\Omega; \Lambda^k)$, then we have a gain of regularity for the traces. More precisely, we have $\text{Tr}_T(\omega) \in W^{1-\frac{1}{p},p}(\Omega; \Lambda^{k+1})$ and $\text{Tr}_N(\omega) \in W^{1-\frac{1}{p},p}(\Omega; \Lambda^{k-1})$, and furthermore,

$$\text{Tr}_T(\omega) = \nu \wedge \text{Tr}(\omega) \text{ and } \text{Tr}_N(\omega) = \nu \lrcorner \text{Tr}(\omega) \text{ on } \partial\Omega.$$

2. For smooth (up to the boundary) differential forms, the tangential and normal components defined here coincides with the usual definition (see [20] or [21] for a detailed discussion on tangential and normal components for classical differential forms) tangential component $\omega_T := \nu \wedge \omega$ and normal component $\omega_N := \nu \lrcorner \omega$ respectively.

3. Moreover,

$$\begin{aligned} W_T^{d,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{d,p}(\Omega; \Lambda^k) : \text{Tr}_T(\omega) = 0 \text{ on } \partial\Omega \right\}. \\ W_N^{\delta,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{\delta,p}(\Omega; \Lambda^k) : \text{Tr}_N(\omega) = 0 \text{ on } \partial\Omega \right\}. \end{aligned}$$

4. From here onwards we shall use the notations $\nu \wedge \omega$ and $\nu \lrcorner \omega$ to mean tangential and normal trace respectively.

Admissible boundary coordinates

Another important notion concerning traces is the notion of admissible boundary coordinates. This will be an indispensable tool for regularity theory later, to flatten the boundary.

Definition 2.31 Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $x_0 \in \partial\Omega$ and let $V \subset \mathbb{R}^n$ be an open neighborhood of x_0 in \mathbb{R}^n . We say the map $\Phi : U \rightarrow V$ is an admissible (local) boundary coordinate system for Ω around x_0 if

1. $U \subset \mathbb{R}^n$ is an open neighborhood of $(y'_0, 0)$ for some $y'_0 \in \mathbb{R}^{n-1}$,
2. $\Phi((y'_0, 0)) = x_0$,
3. $\Phi(U) = V$,
4. $\partial\Omega \cap V = \{\Phi((y', 0)) : (y', 0) \in U\}$, $V \cap \Omega = \{\Phi(y', y_n) : (y', y_n) \in U \text{ and } y_n < 0\}$,
5. For every $1 \leq i \leq n$ and for every $(y', 0) \in U$,

$$\left\langle \frac{\partial \Phi}{\partial y_i}(y', 0); \frac{\partial \Phi}{\partial y_n}(y', 0) \right\rangle = \delta_{in},$$

where δ denotes the Kronecker delta.

For any open set $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega$ is of class $C^{r,\alpha}$ for some integer $r \geq 1$ and some $0 \leq \alpha \leq 1$, then for any $x_0 \in \partial\Omega$, there exist an open set $U \subset \mathbb{R}^n$, an open neighborhood $V \subset \mathbb{R}^n$ of x_0 in \mathbb{R}^n and an admissible boundary coordinate system $\Phi \in \text{Diff}^{r,\alpha}(U; V)$. See Proposition 3.17 in [21] or [53] for a proof.

The importance of an admissible boundary coordinate system is that it helps us to reduce the vanishing of tangential and normal components at the boundary to particularly simple forms.

Proposition 2.32 *Let $\Omega \subset \mathbb{R}^n$ be an open C^2 set and let $0 \leq k \leq n$ be an integer. Let $\omega \in W^{1,p}(\Omega; \Lambda^k)$ for some $1 \leq p < \infty$ and let $x_0 \in \partial\Omega$ and $\Phi : U \rightarrow V$ is an admissible (local) boundary coordinate system for Ω around x_0 . We set $\beta = \Phi^*(\omega)$. Then we have the following:*

1. $\nu \wedge \omega = 0$ on $V \cap \partial\Omega$ if and only if $\beta_{i_1 \dots i_k} = 0$ on $U \cap \partial\mathbb{R}_+^n$ for every $1 \leq i_1 < \dots < i_k < n$.
2. $\nu \lrcorner \omega = 0$ on $V \cap \partial\Omega$ if and only if $\beta_{i_1 \dots i_k} = 0$ on $U \cap \partial\mathbb{R}_+^n$ for every $1 \leq i_1 < \dots < i_{k-1} < i_k = n$.
3. $\nu \wedge \omega = 0$ on $V \cap \partial\Omega$ implies $\nu \wedge d\omega = 0$ on $V \cap \partial\Omega$.
4. $\nu \lrcorner \omega = 0$ on $V \cap \partial\Omega$ implies $\nu \lrcorner d\omega = 0$ on $V \cap \partial\Omega$.

The proof of this result for the case of smooth differential forms can be found in Section 3.2 in [21] (cf. Corollary 3.21 and Theorem 3.23 in particular). By continuity of the trace map the result holds in the $W^{1,p}$ setting as well via density.

Now we need a few important subspaces.

Definition 2.33 *Let $r \geq 0$ is an integer. The spaces $C_T^r(\overline{\Omega}; \Lambda^k)$ and $C_N^r(\overline{\Omega}; \Lambda^k)$ are defined by,*

$$C_T^r(\overline{\Omega}; \Lambda^k) := \left\{ \omega \in C^r(\overline{\Omega}; \Lambda^k) : \nu \wedge \omega = 0 \text{ on } \partial\Omega \right\}$$

and

$$C_N^r(\overline{\Omega}; \Lambda^k) := \left\{ \omega \in C^r(\overline{\Omega}; \Lambda^k) : \nu \lrcorner \omega = 0 \text{ on } \partial\Omega \right\}.$$

Now we state a density result. See [40] for the proof.

Theorem 2.34 *Let $r \geq 1$ is an integer, $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open C^{r+1} set. Then the following statements hold true.*

- (i) $C_T^r(\overline{\Omega}; \Lambda^k)$ is dense in $W_T^{d,p}(\Omega; \Lambda^k)$.
- (ii) $C_c^r(\Omega; \Lambda^k)$ is dense in $W_T^{d,p}(\Omega; \Lambda^k)$.
- (iii) $C_N^r(\overline{\Omega}; \Lambda^k)$ is dense in $W_N^{\delta,p}(\Omega; \Lambda^k)$.
- (iv) $C_c^r(\overline{\Omega}; \Lambda^k)$ is dense in $W_N^{\delta,p}(\Omega; \Lambda^k)$.

We also record the Gauss-Green theorem and Integration by parts formula below. Again, the case of smooth differential forms is easy and the results follow by density.

Theorem 2.35 (Gauss-Green theorem) *Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and let $1 < p < \infty$. Then the following holds.*

- If $\omega \in W^{d,p}(\Omega; \Lambda^k)$, then,

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \nu \wedge \omega.$$

- If $\omega \in W^{\delta,p}(\Omega; \Lambda^k)$, then,

$$\int_{\Omega} \delta\omega = \int_{\partial\Omega} \nu \lrcorner \omega.$$

Theorem 2.36 (Integration by parts formula) *Let $1 \leq k \leq n-1$ and let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and let $\alpha \in W^{1,p}(\Omega; \Lambda^k)$ and $\beta \in W^{1,q}(\Omega; \Lambda^k)$. Then*

$$\int_{\Omega} \langle d\alpha; \beta \rangle + \int_{\Omega} \langle \alpha; \delta\beta \rangle = \int_{\partial\Omega} \langle \nu \wedge \alpha; \beta \rangle = \int_{\partial\Omega} \langle \alpha; \nu \lrcorner \beta \rangle.$$

Remark 2.37 *If on the formula above, if $\alpha \in W^{d,p}(\Omega; \Lambda^k)$ and $\beta \in W^{1,q}(\Omega; \Lambda^k)$, then we still have*

$$\int_{\Omega} \langle d\alpha; \beta \rangle + \int_{\Omega} \langle \alpha; \delta\beta \rangle = \int_{\partial\Omega} \langle \nu \wedge \alpha; \beta \rangle.$$

Similarly, if $\alpha \in W^{1,p}(\Omega; \Lambda^k)$ and $\beta \in W^{\delta,q}(\Omega; \Lambda^k)$, we have

$$\int_{\Omega} \langle d\alpha; \beta \rangle + \int_{\Omega} \langle \alpha; \delta\beta \rangle = \int_{\partial\Omega} \langle \alpha; \nu \lrcorner \beta \rangle.$$

In both cases, the other boundary integral is not well defined (see Theorems 2.26 and 2.27).

2.4.3 Gaffney inequality and Harmonic fields

We start with the well known Gaffney inequality, the proof of which is well known and hence omitted here and can be found, among other places, in theorem 4.8 in [40] and theorem 5.16 in [21].

Theorem 2.38 (Gaffney Inequality) *Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Then there exists a constant $C_p = C_p(\Omega)$ such that,*

$$\|\omega\|_{W^{1,p}} \leq C_p (\|\omega\|_{L^p} + \|d\omega\|_{L^p} + \|\delta\omega\|_{L^p})$$

for every $\omega \in W_T^{1,p}(\Omega; \Lambda^k) \cup W_N^{1,p}(\Omega; \Lambda^k)$.

Now we need the notion of *harmonic fields*.

Definition 2.39 (Harmonic fields) *Let $0 \leq k \leq n$ be an integer and $\Omega \subset \mathbb{R}^n$ be an open set. The space of harmonic k -fields on Ω is defined by,*

$$\mathcal{H}(\Omega; \Lambda^k) := \{\omega \in W^{1,2}(\Omega; \Lambda^k) : d\omega = 0 \text{ and } \delta\omega = 0\}.$$

If $\partial\Omega$ is regular enough, we define the space of harmonic k -fields with vanishing tangential component on $\partial\Omega$ and the space of harmonic k -fields with vanishing normal component on $\partial\Omega$ on Ω , respectively, by the following:

$$\mathcal{H}_T(\Omega; \Lambda^k) := \{\omega \in \mathcal{H}(\Omega; \Lambda^k) : \nu \wedge \omega = 0 \text{ on } \partial\Omega\},$$

and

$$\mathcal{H}_N(\Omega; \Lambda^k) := \{\omega \in \mathcal{H}(\Omega; \Lambda^k) : \nu \lrcorner \omega = 0 \text{ on } \partial\Omega\}.$$

Clearly, if $\partial\Omega$ is regular enough, all these space $\mathcal{H}(\Omega; \Lambda^k)$, $\mathcal{H}_T(\Omega; \Lambda^k)$ and $\mathcal{H}_N(\Omega; \Lambda^k)$ are closed subspaces of the Hilbert space $L^2(\Omega; \Lambda^k)$ and hence have orthogonal complements in L^2 . We denote the complements of these spaces in L^2 by $\mathcal{H}^\perp(\Omega; \Lambda^k)$, $\mathcal{H}_T^\perp(\Omega; \Lambda^k)$ and $\mathcal{H}_N^\perp(\Omega; \Lambda^k)$ respectively. Thus we have the following direct sum decompositions which are orthogonal with respect to the L^2 inner product:

$$\begin{aligned} L^2(\Omega; \Lambda^k) &= \mathcal{H}_T(\Omega; \Lambda^k) \oplus \mathcal{H}_T^\perp(\Omega; \Lambda^k), \\ L^2(\Omega; \Lambda^k) &= \mathcal{H}_N(\Omega; \Lambda^k) \oplus \mathcal{H}_N^\perp(\Omega; \Lambda^k), \end{aligned}$$

and

$$L^2(\Omega; \Lambda^k) = \mathcal{H}(\Omega; \Lambda^k) \oplus \mathcal{H}^\perp(\Omega; \Lambda^k).$$

An immediate corollary of Gaffney inequality for harmonic fields is that the spaces $\mathcal{H}_T(\Omega; \Lambda^k)$ and $\mathcal{H}_N(\Omega; \Lambda^k)$ are always finite dimensional. Indeed, Gaffney inequality implies that for any $h \in \mathcal{H}_T(\Omega; \Lambda^k)$ (or $\mathcal{H}_N(\Omega; \Lambda^k)$) we have,

$$\|h\|_{W^{1,p}} \leq c \|h\|_{L^p}$$

for any $1 < p < \infty$. Now since the embedding

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow L^p(\Omega; \Lambda^k)$$

is compact, this implies that the closed unit ball in $\mathcal{H}_T(\Omega; \Lambda^k)$ (or $\mathcal{H}_N(\Omega; \Lambda^k)$) is compact, implying the finite dimensionality.

Also, since every harmonic field $h \in \mathcal{H}(\Omega; \Lambda^k)$ satisfies

$$\Delta h = \delta dh + d\delta h = 0,$$

we immediately obtain from classical Weyl's lemma that every harmonic field is C^∞ in the interior of the domain. Another well known facts about harmonic fields is that if $h \in \mathcal{H}_T(\Omega; \Lambda^k)$ (or $\mathcal{H}_N(\Omega; \Lambda^k)$) and $\partial\Omega$ is regular enough, then $h \in C^\infty(\bar{\Omega}; \Lambda^k)$. We shall prove even more general up to the boundary regularity results later in the second part of the thesis, so for now we omit the proof. We shall also mention a well known result about these spaces, which is a

special case of the classical deRham theory.

Proposition 2.40 *Let $\Omega \subset \mathbb{R}^n$ be such that there exist $x_0 \in \Omega$ and $F \in C^\infty([0, 1] \times \Omega; \Omega)$ such that for every $x \in \Omega$,*

$$F(0, x) = x_0 \quad \text{and} \quad F(1, x) = x.$$

Such a set Ω is called a contractible set.

Then if moreover Ω is bounded, open, C^2 set, we have,

$$\mathcal{H}_T(\Omega; \Lambda^k) = \{0\} \quad \text{if } 0 \leq k \leq n-1,$$

and

$$\mathcal{H}_N(\Omega; \Lambda^k) = \{0\} \quad \text{if } 1 \leq k \leq n.$$

The proof can be found in [21] (cf. Theorem 6.5). It uses only classical Poincaré lemma and does not use the Hodge decomposition theorem. With classical Poincaré lemma, since for any $h \in \mathcal{H}_N(\Omega; \Lambda^k)$, $dh = 0$ and Ω is contractible, we can find a $k-1$ -form g such that $h = dg$ in Ω . Hence, we have, integrating by parts and using the fact that $\nu \lrcorner h = 0$ on $\partial\Omega$,

$$\|h\|_{L^2}^2 = \int_{\Omega} \langle h; h \rangle = \int_{\Omega} \langle dg; h \rangle = - \int_{\Omega} \langle g; \delta h \rangle + \int_{\partial\Omega} \langle g; \nu \lrcorner h \rangle = 0.$$

This implies $h = 0$. The proof for $\mathcal{H}_T(\Omega; \Lambda^k)$ follows by duality.

Now we record another corollary of Gaffney inequality. For the proof, see theorem 4.11 in [40].

Corollary 2.41 *Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Then there exists a constant $C_p = C_p(\Omega)$ such that,*

$$\|\omega\|_{W^{1,p}} \leq C_p (\|d\omega\|_{L^p} + \|\delta\omega\|_{L^p})$$

for every $\omega \in W_T^{1,p}(\Omega; \Lambda^k) \cap \mathcal{H}_T^\perp(\Omega; \Lambda^k)$. Also the same holds true if $\omega \in W_N^{1,p}(\Omega; \Lambda^k) \cap \mathcal{H}_N^\perp(\Omega; \Lambda^k)$ instead.

2.5 Decomposition theorems and consequences

2.5.1 Hodge-Morrey decomposition

We state the classical Hodge-Morrey decomposition in this subsection. The theorem is well-known and we do not include a proof here (cf. theorem 6.12 in [21], also [40], [53], [58], [68]).

Theorem 2.42 (Hodge-Morrey decomposition) *Let $r \geq 0$ and $0 \leq k \leq n$ be integers and let $0 < \alpha < 1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth set with exterior unit normal ν . Let $f \in W^{r,p}(\Omega, \Lambda^k)$, respectively $f \in C^{r,\alpha}(\bar{\Omega}, \Lambda^k)$.*

(i) *There exist*

$$\alpha \in W_T^{r+1,p}(\Omega, \Lambda^{k-1}), \quad \beta \in W_T^{r+1,p}(\Omega, \Lambda^{k+1}),$$

$$h \in \mathcal{H}_T(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in W_T^{r+2,p}(\Omega, \Lambda^k),$$

respectively

$$\alpha \in C_T^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}), \quad \beta \in C_T^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k+1}),$$

$$h \in \mathcal{H}_T(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in C_T^{r+2,\alpha}(\overline{\Omega}, \Lambda^k),$$

such that in Ω , we have,

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega \quad \text{and} \quad \beta = d\omega.$$

Moreover, there exist constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that

$$\|\omega\|_{W^{r+2,p}} + \|h\|_{W^{r,p}} \leq C_1 \|f\|_{W^{r,p}}$$

$$\|\omega\|_{C^{r+2,\alpha}} + \|h\|_{C^{r,\alpha}} \leq C_2 \|f\|_{C^{r,\alpha}}.$$

(ii) *There exist*

$$\alpha \in W_N^{r+1,p}(\Omega, \Lambda^{k-1}), \quad \beta \in W_N^{r+1,p}(\Omega, \Lambda^{k+1}),$$

$$h \in \mathcal{H}_N(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in W_N^{r+2,p}(\Omega, \Lambda^k),$$

respectively

$$\alpha \in C_N^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}), \quad \beta \in C_N^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k+1}),$$

$$h \in \mathcal{H}_N(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in C_N^{r+2,\alpha}(\overline{\Omega}, \Lambda^k),$$

such that in Ω , we have,

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega \quad \text{and} \quad \beta = d\omega.$$

Moreover, there exist constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that

$$\|\omega\|_{W^{r+2,p}} + \|h\|_{W^{r,p}} \leq C_1 \|f\|_{W^{r,p}}$$

$$\|\omega\|_{C^{r+2,\alpha}} + \|h\|_{C^{r,\alpha}} \leq C_2 \|f\|_{C^{r,\alpha}}.$$

(iii) *There exist*

$$\alpha \in W_T^{r+1,p}(\Omega, \Lambda^{k-1}), \quad \beta \in W_N^{r+1,p}(\Omega, \Lambda^{k+1}),$$

$$h \in \mathcal{H}(\Omega; \Lambda^k), \quad \omega^1 \in W_T^{r+2,p}(\Omega, \Lambda^k) \quad \text{and} \quad \omega^2 \in W_N^{r+2,p}(\Omega, \Lambda^k),$$

respectively

$$\alpha \in C_T^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}), \beta \in C_N^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k+1}),$$

$$h \in \mathcal{H}(\Omega; \Lambda^k), \omega^1 \in C_T^{r+2,\alpha}(\overline{\Omega}, \Lambda^k) \text{ and } \omega^2 \in C_N^{r+2,p}(\overline{\Omega}, \Lambda^k),$$

such that in Ω , we have,

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega^1 \text{ and } \beta = d\omega^2.$$

Moreover, there exist constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that

$$\|\omega^1\|_{W^{r+2,p}} + \|\omega^2\|_{W^{r+2,p}} + \|h\|_{W^{r,p}} \leq C_1 \|f\|_{W^{r,p}}$$

$$\|\omega^1\|_{C^{r+2,\alpha}} + \|\omega^2\|_{C^{r+2,\alpha}} + \|h\|_{C^{r,\alpha}} \leq C_2 \|f\|_{C^{r,\alpha}}.$$

2.5.2 Classical boundary value problems for differential forms

We now show the solvability of certain boundary value problems ((\mathcal{P}_T) and (\mathcal{P}_d) below) involving differential forms, which is crucial to settle minimization problems (e.g. Theorem 3.64, Theorem 3.67). The results are already known and are proved for the restricted case $2 \leq p < \infty$ in [21] (cf. theorem 7.2 and 8.16 in [21]) and [58]. Essentially both the results follow from Hodge-Morrey decomposition (theorem 2.42). But the methods presented in [21] can be extended to the case $1 < p < \infty$ with slight modification of the argument. The aforementioned modification essentially amounts to arguing via L^p - $L^{p'}$ duality instead of the L^2 norm. Also, since apart from this modification, the proof is essentially the same, we prove only one of the theorems presented below to illustrate the modification.

The first one is a generalized div-curl type systems, sometimes called a Cauchy-Riemann type systems.

Theorem 2.43 (Div-Curl Systems with tangential data) *Let $r \geq 0$ and $0 \leq k \leq n$ be integers. Let $0 < \alpha < 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν . Let $f : \overline{\Omega} \rightarrow \Lambda^{k+1}$, $g : \overline{\Omega} \rightarrow \Lambda^{k-1}$ and $\omega_0 : \partial\Omega \rightarrow \Lambda^k$. Then the following statements are equivalent:*

(i) *Let*

$$f \in W^{r,p}(\Omega; \Lambda^{k+1}), \quad g \in W^{r,p}(\Omega; \Lambda^{k-1}) \text{ and } \nu \wedge \omega_0 \in W^{r+1-\frac{1}{p},p}(\partial\Omega; \Lambda^{k+1}),$$

respectively

$$f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1}), \quad g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k-1}) \text{ and } \nu \wedge \omega_0 \in C^{r+1,\alpha}(\partial\Omega; \Lambda^{k+1}),$$

satisfy the conditions

$$df = 0 \text{ in } \Omega, \quad \delta g = 0 \text{ in } \Omega, \text{ and } \nu \wedge d\omega_0 = \nu \wedge f \text{ on } \partial\Omega, \quad (\text{A1})$$

and for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$ and $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$,

$$\int_{\Omega} \langle f; \chi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle = 0 \text{ and } \int_{\Omega} \langle g; \psi \rangle = 0. \quad (\text{A2})$$

(ii) There exists $\omega \in W^{r+1,p}(\Omega; \Lambda^k)$, respectively $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, such that

$$\begin{cases} d\omega = f & \text{and } \delta\omega = g & \text{in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 & & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_T)$$

In addition, there exist positive constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that,

$$\|\omega\|_{W^{r+1,p}(\Omega)} \leq C_1 \left(\|f\|_{W^{r,p}(\Omega)} + \|g\|_{W^{r,p}(\Omega)} + \|\nu \wedge \omega_0\|_{W^{r+1-\frac{1}{p},p}(\partial\Omega)} \right),$$

respectively

$$\|\omega\|_{C^{r+1,\alpha}(\Omega)} \leq C_2 \left(\|f\|_{C^{r,\alpha}(\Omega)} + \|g\|_{C^{r,\alpha}(\Omega)} + \|\nu \wedge \omega_0\|_{C^{r+1,\alpha}(\partial\Omega)} \right).$$

Remark 2.44 When $r = 0$, the condition $\nu \wedge d\omega_0 = \nu \wedge f$ on $\partial\Omega$ in (A1) is to be interpreted as,

$$\int_{\Omega} \langle f; \delta\phi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \delta\phi \rangle = 0$$

for every $\phi \in C^\infty(\overline{\Omega}, \Lambda^{k+2})$. See remark 7.3(iii) in [21] for details.

Proof We only prove the the Sobolev case to illustrate how we can remove the restriction $p \geq 2$ in the proof of theorem 7.2 in [21] (see also [58]). We also assume $r = 0$ to show how to tackle the ‘weak’ form of the condition $\nu \wedge d\omega_0 = \nu \wedge f$ on $\partial\Omega$.

(ii) \Rightarrow (i): The first two conditions in (A1) follows by integrating by parts , since

$$\int_{\Omega} \langle f; \delta\varphi \rangle = \int_{\Omega} \langle d\omega; \delta\varphi \rangle = - \int_{\Omega} \langle \omega; \delta\delta\varphi \rangle = 0$$

for every $\varphi \in C_0^\infty(\Omega; \Lambda^{k+2})$, and we also have,

$$\int_{\Omega} \langle g; d\varphi \rangle = \int_{\Omega} \langle \delta\omega; d\varphi \rangle = - \int_{\Omega} \langle \omega; dd\varphi \rangle = 0$$

for every $\varphi \in C_0^\infty(\Omega; \Lambda^{k-2})$. For the third condition in (A1), we have, by integrating by parts,

$$\int_{\Omega} \langle f; \delta\phi \rangle = \int_{\Omega} \langle d\omega; \delta\phi \rangle = - \int_{\Omega} \langle \omega; \delta\delta\phi \rangle + \int_{\partial\Omega} \langle \nu \wedge \omega; \delta\phi \rangle = \int_{\partial\Omega} \langle \nu \wedge \omega_0; \delta\phi \rangle$$

for every $\phi \in C^\infty(\overline{\Omega}, \Lambda^{k+2})$.

The first condition in (A2) follows by integrating by parts. Indeed, for any $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$,

$$\begin{aligned} \int_{\Omega} \langle f; \chi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle &= \int_{\Omega} \langle d\omega; \chi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle \\ &= \int_{\partial\Omega} \langle \nu \wedge (\omega - \omega_0); \chi \rangle - \int_{\Omega} \langle \omega; \delta\chi \rangle = 0. \end{aligned}$$

The second condition in (A2) follows in a similar way. We have,

$$\int_{\Omega} \langle g; \psi \rangle = \int_{\Omega} \langle \delta\omega; \psi \rangle = - \int_{\Omega} \langle \omega; d\psi \rangle + \int_{\partial\Omega} \langle \omega; \nu \wedge \psi \rangle = 0$$

for every $\psi \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$.

(i) \Rightarrow (ii): We first extend (see Lemma 7.1 of [21]) ω_0 by $\tilde{\omega}_0$ to the full domain Ω so that $\tilde{\omega}_0 \in W^{1,p}(\Omega; \Lambda^k)$ and $\nu \wedge \omega_0 = \nu \wedge \tilde{\omega}_0$ on $\partial\Omega$ and there is a constant $c = c(\Omega, p)$ such that

$$\|\tilde{\omega}_0\|_{W^{1,p}(\Omega; \Lambda^k)} \leq c \|\omega_0\|_{W^{1-\frac{1}{p},p}(\partial\Omega; \Lambda^k)}.$$

Step 1 We now show that (A1) implies the following two equations

$$\int_{\Omega} \langle f; \delta\varphi \rangle - \int_{\Omega} \langle d\tilde{\omega}_0; \delta\varphi \rangle = 0 \quad \forall \varphi \in C^\infty(\overline{\Omega}; \Lambda^{k+2}) \quad (2.1)$$

$$\int_{\Omega} \langle g; d\psi \rangle = 0 \quad \forall \psi \in C_T^\infty(\overline{\Omega}; \Lambda^{k-2}). \quad (2.2)$$

Since $d\tilde{\omega}_0$ are closed in the sense of distributions, we have,

$$\int_{\Omega} \langle f; \delta\varphi \rangle = \int_{\partial\Omega} \langle \nu \wedge \omega_0; \delta\phi \rangle = \int_{\Omega} \langle d\tilde{\omega}_0; \delta\varphi \rangle.$$

Equation (2.2) follows immediately from the second equation in (A1).

Step 2 We apply the Hodge-Morrey decomposition(cf. Theorem 6.12 in [21]) to decompose $f - d\tilde{\omega}_0$ and obtain (if $k = n$, we do not need this construction),

$$\begin{aligned} f - d\tilde{\omega}_0 &= d\alpha_f + \delta\beta_f + \chi_f \quad \text{in } \Omega \\ \delta\alpha_f &= 0, \quad d\beta_f = 0 \quad \text{in } \Omega \\ \nu \wedge \alpha_f &= 0, \quad \nu \wedge \beta_f = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\chi_f \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$. Moreover there exists a positive constant $C = C(p, \Omega)$ such that

$$\|\alpha_f\|_{W^{1,p}(\overline{\Omega})} \leq C \left(\|f\|_{L^p(\overline{\Omega})} + \|\tilde{\omega}_0\|_{W^{1,p}(\overline{\Omega})} \right).$$

We claim that $\delta\beta_f$ and χ_f vanish. Firstly, since χ_f is a harmonic field, χ_f is C^∞ and hence $\chi_f \in L^p$ for all $1 < p < \infty$ and also $d\chi_f = 0 = \delta\chi_f$, we have, using condition (2.1) and

integrating by parts,

$$\begin{aligned}
0 &= \int_{\Omega} \langle \chi_f; f \rangle - \int_{\partial\Omega} \langle \chi_f; \nu \wedge \omega_0 \rangle = \int_{\Omega} \langle \chi_f; f - d\tilde{\omega}_0 \rangle = \int_{\Omega} \langle \chi_f; d\alpha_f + \delta\beta_f + \chi_f \rangle \\
&= \int_{\Omega} \langle \chi_f; d\alpha_f \rangle + \int_{\Omega} \langle \chi_f; \delta\beta_f \rangle + \int_{\Omega} \langle \chi_f; \chi_f \rangle = - \int_{\Omega} \langle \delta\chi_f; \alpha_f \rangle - \int_{\Omega} \langle d\chi_f; \beta_f \rangle + \int_{\Omega} |\chi_f|^2 \\
&= \int_{\Omega} |\chi_f|^2.
\end{aligned}$$

This implies $\chi_f = 0$. Now we have,

$$\begin{aligned}
0 &= \int_{\Omega} \langle f - d\tilde{\omega}_0; \delta\varphi \rangle = \int_{\Omega} \langle d\alpha_f; \delta\varphi \rangle + \int_{\Omega} \langle \delta\beta_f; \delta\varphi \rangle = - \int_{\Omega} \langle \alpha_f; \delta\delta\varphi \rangle + \int_{\Omega} \langle \delta\beta_f; \delta\varphi \rangle \\
&= \int_{\Omega} \langle \delta\beta_f; \delta\varphi \rangle,
\end{aligned}$$

for every $\varphi \in C^\infty(\overline{\Omega}; \Lambda^{k+2})$. This implies $\delta\beta_f = 0$ by virtue of density of $\delta C^\infty(\overline{\Omega}; \Lambda^{k+2})$ in $\delta W^{1,p'}(\overline{\Omega}; \Lambda^{k+2})$, which is the dual of $\delta W^{1,p}(\overline{\Omega}; \Lambda^{k+2})$ (cf. [40] for these and lots more related density and duality results). Hence we have found $\alpha_f \in W^{1,p}(\overline{\Omega}; \Lambda^k)$ satisfying (if $k = n$, we take $\alpha_f = 0$)

$$\begin{cases} d\alpha_f = f - d\tilde{\omega}_0 & \text{and} & \delta\alpha_f = 0 & \text{in } \Omega \\ \nu \wedge \alpha_f = 0 & & & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

We now apply the same decomposition to $g - \delta\tilde{\omega}_0$ (if $k = 0$, we do not need this construction) and get

$$\begin{aligned}
g - \delta\tilde{\omega}_0 &= d\alpha_g + \delta\beta_g + \psi_g & \text{in } \Omega \\
\delta\alpha_g &= 0, \quad d\beta_g = 0 & \text{in } \Omega \\
\nu \wedge \alpha_g &= 0, \quad \nu \wedge \beta_g = 0 & \text{on } \partial\Omega,
\end{aligned}$$

where $\psi_g \in \mathcal{H}_T(\Omega; \Lambda^{k-1})$. Moreover there exists a positive constant $C = C(p, \Omega)$ such that

$$\|\beta_g\|_{W^{1,p}(\overline{\Omega})} \leq C \left(\|g\|_{L^p(\overline{\Omega})} + \|\tilde{\omega}_0\|_{C^{r+1,q}(\overline{\Omega})} \right).$$

Using (2.2), the second equation in (A2), and the similar argument as before, we have that $d\alpha_g$ and ψ_g vanish. Hence we have found $\beta_g \in C^{r+1,q}(\Omega; \Lambda^k)$ satisfying (if $k = 0$, we take $\beta_g = 0$)

$$\begin{cases} d\beta_g = 0 & \text{and} & \delta\beta_g = g - \delta\tilde{\omega}_0 & \text{in } \Omega \\ \nu \wedge \beta_g = 0 & & & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

We now set

$$\omega = \alpha_f + \beta_g + \tilde{\omega}_0$$

which satisfies, due to (2.3) and (2.4),

$$\begin{cases} d\omega = d\alpha_f + d\tilde{\omega}_0 = f & \text{and} & \delta\omega = \delta\beta_g + \delta\tilde{\omega}_0 = g & \text{in } \Omega \\ \nu \wedge \omega = \nu \wedge \tilde{\omega}_0 = \nu \wedge \omega_0 & & & \text{on } \partial\Omega. \end{cases}$$

This concludes the proof. ■

Remark 2.45 *The modification in the proof is applicable in the same way to several results presented in [21] that has the restriction $p \geq 2$ (see also [58]). Comparing the proof of the above theorem presented here and the proof of theorem 7.2 in [21], one easily sees that the basic point is, if $1 < p < 2$, writing expressions like $\int_{\Omega} |\delta\beta_f|^2$ or $\int_{\Omega} |d\alpha_f|^2$ is no longer possible, as they do not make sense (though $\int_{\Omega} |\chi_f|^2$ is well defined since χ_f , being a harmonic field is C^∞). The trick is to argue instead with expressions like $\int_{\Omega} \langle \delta\beta_f; \delta a \rangle$ for δa in the dual space of $\delta\beta_f$.*

Now we present the result for normal boundary data.

Theorem 2.46 (Div-Curl Systems with normal data) *Let $r \geq 0$ and $0 \leq k \leq n$ be integers. Let $0 < \alpha < 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν . Let $f : \bar{\Omega} \rightarrow \Lambda^{k+1}$, $g : \bar{\Omega} \rightarrow \Lambda^{k-1}$ and $\omega_0 : \partial\Omega \rightarrow \Lambda^k$. Then the following statements are equivalent:*

(i) *Let*

$$f \in W^{r,p}(\Omega; \Lambda^{k+1}), \quad g \in W^{r,p}(\Omega; \Lambda^{k-1}) \quad \text{and} \quad \nu \lrcorner \omega_0 \in W^{r+1-\frac{1}{p},p}(\partial\Omega; \Lambda^{k+1}),$$

respectively

$$f \in C^{r,\alpha}(\bar{\Omega}; \Lambda^{k+1}), \quad g \in C^{r,\alpha}(\bar{\Omega}; \Lambda^{k-1}) \quad \text{and} \quad \nu \lrcorner \omega_0 \in C^{r+1,\alpha}(\partial\Omega; \Lambda^{k+1}),$$

satisfy the conditions

$$df = 0 \text{ in } \Omega, \quad \delta g = 0 \text{ in } \Omega, \quad \text{and} \quad \nu \lrcorner \delta \omega_0 = \nu \lrcorner g \text{ on } \partial\Omega, \quad (\text{A1})$$

and for every $\chi \in \mathcal{H}_N(\Omega; \Lambda^{k+1})$ and $\psi \in \mathcal{H}_N(\Omega; \Lambda^{k-1})$,

$$\int_{\Omega} \langle g; \chi \rangle - \int_{\partial\Omega} \langle \nu \lrcorner \omega_0; \chi \rangle = 0 \quad \text{and} \quad \int_{\Omega} \langle f; \psi \rangle = 0. \quad (\text{A2})$$

(ii) *There exists $\omega \in W^{r+1,p}(\Omega; \Lambda^k)$, respectively $\omega \in C^{r+1,\alpha}(\bar{\Omega}; \Lambda^k)$, such that*

$$\begin{cases} d\omega = f & \text{and} & \delta\omega = g & \text{in } \Omega, \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 & & & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_N)$$

In addition, there exist positive constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that,

$$\|\omega\|_{W^{r+1,p}(\Omega)} \leq C_1 \left(\|f\|_{W^{r,p}(\Omega)} + \|g\|_{W^{r,p}(\Omega)} + \|\nu \wedge \omega_0\|_{W^{r+1-\frac{1}{p},p}(\partial\Omega)} \right),$$

respectively

$$\|\omega\|_{C^{r+1,\alpha}(\Omega)} \leq C_2 \left(\|f\|_{C^{r,\alpha}(\Omega)} + \|g\|_{C^{r,\alpha}(\Omega)} + \|\nu \wedge \omega_0\|_{C^{r+1,\alpha}(\partial\Omega)} \right).$$

We now present the other theorems without proof. The proofs for $p \geq 2$ are in [21] and the same modification as above removes this restriction.

Theorem 2.47 (Poincaré lemma for d with Dirichlet data) *Let $r \geq 0$ and $0 \leq k \leq n$ be integers. Let $0 < \alpha < 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν . Let $f : \overline{\Omega} \rightarrow \Lambda^{k+1}$, $g : \overline{\Omega} \rightarrow \Lambda^{k-1}$ and $\omega_0 : \partial\Omega \rightarrow \Lambda^k$. Then the following statements are equivalent:*

- (i) *Let $f \in W^{r,p}(\Omega; \Lambda^{k+1})$ and $\omega_0 \in W^{r+1-\frac{1}{p},p}(\partial\Omega; \Lambda^k)$, respectively $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ and $\omega_0 \in C^{r+1,\alpha}(\partial\Omega; \Lambda^k)$, satisfy the conditions*

$$df = 0 \text{ in } \Omega, \quad \nu \wedge d\omega_0 = \nu \wedge f \text{ on } \partial\Omega, \quad (\text{B1})$$

and for every $\chi \in \mathcal{H}_T(\Omega; \Lambda^{k+1})$,

$$\int_{\Omega} \langle f; \chi \rangle - \int_{\partial\Omega} \langle \nu \wedge \omega_0; \chi \rangle = 0. \quad (\text{B3})$$

- (ii) *There exists $\omega \in W^{r+1,p}(\Omega; \Lambda^k)$, respectively $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, such that*

$$\begin{cases} d\omega = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_d)$$

In addition, there exist positive constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that,

$$\|\omega\|_{W^{r+1,p}(\Omega)} \leq C_1 \left(\|f\|_{W^{r,p}(\Omega)} + \|\omega_0\|_{W^{r+1-\frac{1}{p},p}(\partial\Omega)} \right),$$

respectively

$$\|\omega\|_{C^{r+1,\alpha}(\Omega)} \leq C_2 \left(\|f\|_{C^{r,\alpha}(\Omega)} + \|\omega_0\|_{C^{r+1,\alpha}(\partial\Omega)} \right).$$

Theorem 2.48 (Poincaré lemma for δ with Dirichlet data) *Let $r \geq 0$ and $0 \leq k \leq n$ be integers. Let $0 < \alpha < 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν . Let $g : \overline{\Omega} \rightarrow \Lambda^{k-1}$, $g : \overline{\Omega} \rightarrow \Lambda^{k-1}$ and $\omega_0 : \partial\Omega \rightarrow \Lambda^k$. Then the following statements are equivalent:*

- (i) *Let $g \in W^{r,p}(\Omega; \Lambda^{k-1})$ and $\omega_0 \in W^{r+1-\frac{1}{p},p}(\partial\Omega; \Lambda^k)$, respectively $g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k-1})$ and $\omega_0 \in C^{r+1,\alpha}(\partial\Omega; \Lambda^k)$, satisfy the conditions*

$$\delta g = 0 \text{ in } \Omega, \quad \nu \lrcorner \delta \omega_0 = \nu \lrcorner g \text{ on } \partial\Omega, \quad (\text{B1})$$

and for every $\chi \in \mathcal{H}_N(\Omega; \Lambda^{k+1})$,

$$\int_{\Omega} \langle g; \chi \rangle - \int_{\partial\Omega} \langle \nu \lrcorner \omega_0; \chi \rangle = 0. \quad (\text{B3})$$

(ii) There exists $\omega \in W^{r+1,p}(\Omega; \Lambda^k)$, respectively $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$, such that

$$\begin{cases} \delta\omega = g & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_2)$$

In addition, there exist positive constants $C_1 = C_1(r, p, \Omega)$ and $C_2 = C_2(r, \alpha, \Omega)$ such that,

$$\|\omega\|_{W^{r+1,p}(\Omega)} \leq C_1 \left(\|g\|_{W^{r,p}(\Omega)} + \|\omega_0\|_{W^{r+1-\frac{1}{p},p}(\partial\Omega)} \right),$$

respectively

$$\|\omega\|_{C^{r+1,\alpha}(\Omega)} \leq C_2 \left(\|g\|_{C^{r,\alpha}(\Omega)} + \|\omega_0\|_{C^{r+1,\alpha}(\partial\Omega)} \right).$$

2.5.3 Important Consequences

The results in the last subsection immediately imply a number of important results. We start with a few embedding theorems which will be quite useful later. But before stating the result, we need to introduce the following important subspaces.

Definition 2.49 Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $1 \leq p \leq \infty$. We define

$$\begin{aligned} W_{\delta,T}^{d,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W_T^{d,p}(\Omega; \Lambda^k) : \delta\omega = 0 \text{ in the sense of distributions} \right\}, \\ W_{d,N}^{\delta,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W_N^{\delta,p}(\Omega; \Lambda^k) : d\omega = 0 \text{ in the sense of distributions} \right\}. \end{aligned}$$

Now Gaffney inequality implies that these two subspaces actually embed into $W^{1,p}$ for $1 < p < \infty$. This is the content of the following proposition.

Proposition 2.50 Let $1 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $1 < p < \infty$. Then the following continuous embeddings hold,

$$W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \hookrightarrow W^{1,p}(\Omega; \Lambda^k) \quad \text{and} \quad W_{d,N}^{\delta,p}(\Omega; \Lambda^k) \hookrightarrow W^{1,p}(\Omega; \Lambda^k).$$

Moreover, there exist constants $C_{T,p} = C_{T,p}(\Omega)$ and $C_{N,p} = C_{N,p}(\Omega)$ such that,

$$\|\omega\|_{W^{1,p}} \leq C_{T,p} \|d\omega\|_{L^p} \quad \text{for all } \omega \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k),$$

and

$$\|\omega\|_{W^{1,p}} \leq C_{N,p} \|\delta\omega\|_{L^p} \quad \text{for all } \omega \in W_{d,N}^{\delta,p}(\Omega; \Lambda^k).$$

Proof We just prove the first one. The second one is completely analogous. We break the proof in two steps.

Step 1 First we want to show that $\delta\omega = 0$ in the sense distributions implies that

$$\int_{\Omega} \langle \omega; d\phi \rangle = 0 \quad \text{for every } \phi \in W_0^{1,p'}(\Omega; \Lambda^{k-1}), \quad (2.5)$$

where p' is the Hölder conjugate exponent of p . Indeed, by virtue of density of $C_c^\infty(\Omega; \Lambda^{k-1})$ in $W_0^{1,p'}(\Omega; \Lambda^{k-1})$, for any $\phi \in W_0^{1,p'}(\Omega; \Lambda^{k-1})$, we can find a sequence $\{\phi_\varepsilon\} \subset C_c^\infty(\Omega; \Lambda^{k-1})$ such that $d\phi_\varepsilon \rightarrow d\phi$ in $L^{p'}$. Thus, we have,

$$\left| \int_{\Omega} \langle \omega; d\phi - d\phi_\varepsilon \rangle \right| \leq \|\omega\|_{L^p} \|d\phi - d\phi_\varepsilon\|_{L^{p'}} \rightarrow 0.$$

Since $\phi_\varepsilon \in C_c^\infty(\Omega; \Lambda^{k-1})$ and $\delta\omega = 0$ in the sense distributions implies $\int_{\Omega} \langle \omega; d\phi_\varepsilon \rangle = 0$, this shows (2.5).

But (2.5) implies, by definition of weak derivatives, that

$$\int_{\Omega} \langle \delta\omega; \phi \rangle = 0 \quad \text{for every } \phi \in L^{p'}(\Omega; \Lambda^{k-1}),$$

which implies $\delta\omega = 0$ as $L^p(\Omega; \Lambda^{k-1})$ functions and $\|\delta\omega\|_{L^p(\Omega; \Lambda^{k-1})} = 0$.

Step 2 Now we want to show a slightly stronger result than the theorem itself. We shall show that the space $W_T^{d,p}(\Omega; \Lambda^k) \cap W^{\delta,p}(\Omega; \Lambda^k)$ continuously embeds into $W^{1,p}(\Omega; \Lambda^k)$. Since by Step 1, $W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \subset W_T^{d,p}(\Omega; \Lambda^k) \cap W^{\delta,p}(\Omega; \Lambda^k)$, this will imply the proposition.

Now we show the claimed embedding. Since $W^{1,p}(\Omega; \Lambda^k) \cap W_T^{d,p}(\Omega; \Lambda^k)$ is dense in $W_T^{d,p}(\Omega; \Lambda^k)$ (by density of $C_T^\infty(\Omega; \Lambda^k)$ in $W_T^{d,p}$), for every $\omega \in W_T^{d,p}(\Omega; \Lambda^k) \cap W^{\delta,p}(\Omega; \Lambda^k)$, we can find a sequence $\{v_j\} \subset W^{1,p}(\Omega; \Lambda^k) \cap W_T^{d,p}(\Omega; \Lambda^k)$ such that $v_j \rightarrow \omega$ in $W^{d,p}$.

Now using theorem 2.43, we solve, for each j , the following boundary value problem:

$$\begin{cases} du_j = 0 & \text{and } \delta u_j = \delta v_j & \text{in } \Omega, \\ \nu \wedge u_j = 0 & & \text{on } \partial\Omega. \end{cases}$$

Similarly, again by virtue of theorem 2.43, we can solve,

$$\begin{cases} du = 0 & \text{and } \delta u = \delta\omega & \text{in } \Omega, \\ \nu \wedge u = 0 & & \text{on } \partial\Omega. \end{cases}$$

Now we set $\omega_j = v_j - u_j + u$. Now, since $u_j, u \in W^{1,p}(\Omega; \Lambda^k)$, clearly $\omega_j \in W^{1,p}(\Omega; \Lambda^k)$ also. It is also immediate that $\nu \wedge \omega_j = 0$ on $\partial\Omega$. Now, we have,

$$d\omega_j = dv_j - du_j + du = dv_j,$$

and

$$\delta\omega_j = \delta v_j - \delta u_j + \delta u = \delta\omega,$$

for every j . Now by corollary 2.41, we obtain,

$$\begin{aligned}\|\omega_j\|_{W^{1,p}} &\leq c (\|d\omega_j\|_{L^p} + \|\delta\omega_j\|_{L^p}) \\ &= c (\|dv_j\|_{L^p} + \|\delta\omega\|_{L^p}).\end{aligned}$$

But since $v_j \rightarrow \omega$ in $W^{d,p}$, this implies $\{\|\omega_j\|_{W^{1,p}}\}$ is uniformly bounded, since $\{\|v_j\|_{L^p}\}$ is uniformly bounded. Thus, $\omega_j \rightharpoonup \tilde{\omega}$ weakly in $W^{1,p}$ for some $\tilde{\omega} \in W^{1,p}(\Omega; \Lambda^k)$. But since $d\omega_j = dv_j \rightarrow d\omega$ in L^p and $\delta\omega_j = \delta\omega$, by uniqueness of weak limits we have,

$$d\tilde{\omega} = d\omega \quad \text{and} \quad \delta\tilde{\omega} = \delta\omega.$$

Also, $\nu \wedge \tilde{\omega} = 0 = \nu \wedge \omega$ on $\partial\Omega$. Hence, we have,

$$d(\tilde{\omega} - \omega) = 0, \quad \delta(\tilde{\omega} - \omega) = 0 \text{ in } \Omega \text{ and } \nu \wedge (\tilde{\omega} - \omega) = 0 \text{ on } \partial\Omega.$$

This implies $\tilde{\omega} - \omega \in \mathcal{H}_T(\Omega; \Lambda^k)$. Since Ω is contractible, we must have $\tilde{\omega} = \omega$ and this shows $\omega \in W^{1,p}(\Omega; \Lambda^k)$. Continuity of the embedding follows from the estimate obtained by applying corollary 2.41 to ω now. This concludes the proof. ■

Remark 2.51 *Note that the proof actually shows that we have the stronger embeddings:*

$$\begin{aligned}W_T^{d,p}(\Omega; \Lambda^k) \cap W^{\delta,p}(\Omega; \Lambda^k) &\hookrightarrow W^{1,p}(\Omega; \Lambda^k), \\ W_N^{\delta,p}(\Omega; \Lambda^k) \cap W^{d,p}(\Omega; \Lambda^k) &\hookrightarrow W^{1,p}(\Omega; \Lambda^k).\end{aligned}$$

We present a decomposition theorem that will be useful later.

Theorem 2.52 *Let $1 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $1 < p \leq q \leq p^* < \infty$ if $p < n$ or $1 < p \leq q < \infty$ if $p \geq n$. Then there exists a topological direct sum decomposition*

$$W_T^{d,p,q}(\Omega; \Lambda^k) = W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \oplus dW_0^{1,q}(\Omega; \Lambda^{k-1}),$$

where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate exponent of p . Moreover, if $p < n$ and $2 \leq p \leq q \leq p^* < \infty$ or if $p \geq n$ and $2 \leq p \leq q < \infty$, then the decomposition is orthogonal with respect to the L^2 inner product.

Proof First note that if $v \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$, by proposition 2.50, we have $v \in W_T^{1,p}(\Omega; \Lambda^k)$ and hence by Sobolev embedding $v \in L^q(\Omega; \Lambda^k)$, since $q \leq p^*$. Hence $W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \subset W_T^{d,p,q}(\Omega; \Lambda^k)$. Clearly, since Ω has finite measure and $p \leq q$, $dW_0^{1,q}(\Omega; \Lambda^{k-1}) \subset W_T^{d,p,q}(\Omega; \Lambda^k)$ also. Now let $\omega \in W_T^{d,p,q}(\Omega; \Lambda^k)$. Since $\omega \in L^q(\Omega; \Lambda^k)$, by Hodge decomposition theorem there exists $\alpha \in W_T^{1,q}(\Omega; \Lambda^{k-1})$ and $\beta \in W_T^{1,q}(\Omega; \Lambda^{k+1})$ such that

$$\omega = d\alpha + \delta\beta.$$

Now we first show that $\delta\beta \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$.

Indeed, $\delta(\delta\beta) = 0$ in the sense of distributions. Also, since Ω is bounded and $p \leq q$, $\delta\beta \in L^q(\Omega; \Lambda^k) \Rightarrow \delta\beta \in L^p(\Omega; \Lambda^k)$. Also, since $d\omega = d(d\alpha) + d(\delta\beta) = d(\delta\beta)$ and $d\omega \in L^p(\Omega; \Lambda^{k+1})$, we

have $d(\delta\beta) \in L^p(\Omega; \Lambda^k)$, implying $\delta\beta \in W^{d,p}(\Omega; \Lambda^k)$. Again, we have, $0 = \nu \wedge \omega = \nu \wedge d\alpha + \nu \wedge \delta\beta = \nu \wedge \delta\beta$ on $\partial\Omega$, since $\alpha \in W_T^{1,q}(\Omega; \Lambda^{k-1})$ implies $\nu \wedge \alpha = 0$ on $\partial\Omega$, which in turn implies $\nu \wedge d\alpha = 0$ on $\partial\Omega$ (cf. theorem 3.23 in [21]).

Now since $\nu \wedge d\alpha = 0$ on $\partial\Omega$ and $d(d\alpha) = 0$ in the sense of distributions, by theorem 2.47 there exists $\theta \in W^{1,q}(\Omega; \Lambda^{k-1})$ such that,

$$\begin{cases} d\theta = d\alpha & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, we have,

$$\omega = d\theta + \delta\beta,$$

with $\delta\beta \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ and $\theta \in W_0^{1,q}(\Omega; \Lambda^{k-1})$. The decomposition is clearly a direct sum decomposition. The L^2 orthogonality is also obvious. This concludes the proof. ■

Proceeding analogously, we also have the dual statement.

Theorem 2.53 *Let $1 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $1 < p \leq q \leq p^* < \infty$ if $p < n$ or $1 < p \leq q < \infty$ if $p \geq n$. Then there exists a topological direct sum decomposition*

$$W_N^{\delta,p,q}(\Omega; \Lambda^k) = W_{d,N}^{\delta,p}(\Omega; \Lambda^k) \oplus \delta W_0^{1,q}(\Omega; \Lambda^{k-1}),$$

where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate exponent of p . Moreover, if $p < n$ and $2 \leq p \leq q \leq p^* < \infty$ or if $p \geq n$ and $2 \leq p \leq q < \infty$, then the decomposition is orthogonal with respect to the L^2 inner product.

Part I

Direct Methods in Calculus of Variations for Differential Forms

Foreword to part I

The principal aim in this part is to develop a framework for applying the direct methods in calculus of variations to minimization problems involving differential forms. The framework for classical direct methods in vectorial calculus of variations is already well developed and by now, standard. However, as we shall see, the special algebraic features of differential forms demand independent attention.

The main goal in this part is twofold: firstly, *to show that a framework for direct methods for differential forms is indeed possible and can be developed independently of the classical framework*. Such a theoretical pursuit is indeed worthwhile, as we shall also see that the resulting theory is *quite distinct*, i.e contains a lot of features entirely absent from the classical one and is quite rich and interesting in its own right. Secondly, this analysis would also show that in a way, *the language of differential forms is the more natural* of the two frameworks. The determinants and the minors of the Jacobian matrix already play a central role in classical vectorial calculus of variations. We shall put these results into perspective by showing that it is actually the exterior product that should be given this central conceptual role, and determinants and the minors are nothing but particular examples of this general structure.

The material in this part is divided into three chapters. In chapter 3, we shall start carrying out this program of building a framework for direct methods for the case of functionals which depend on exterior (or interior) derivatives of a single differential form. This program will be carried out quite comprehensively, yielding a more or less complete picture in this case. In chapter 4, we shall focus on functionals depending on exterior derivatives of more than one differential forms. Here however, the main focus is the semicontinuity results which generalize the classical semicontinuity theorems in vectorial calculus of variations. These analysis mainly try to make precise the sense in which the language of differential forms should be the more natural one in calculus of variations. We shall indeed take the shortest route to the semicontinuity results. The analysis which we shall undertake for functionals of exterior derivatives of single differential forms will not be carried out completely for functionals depending on several forms. But such an analysis would probably be quite rewarding. We conclude this part with chapter 5, where we discuss the scope of possible generalizations to other type of functionals. Unfortunately, we shall see that the basic results that we can derive already shows us that such generalizations would not yield anything essentially new at the level of ‘quasiaffine’ functions. So we shall confine ourselves mostly to presenting those basic results in chapter 5.

Chapter 3

Functionals depending on exterior derivative of a single differential form

3.1 Introduction

In this chapter we set ourselves the task of developing a framework for applying the so-called ‘direct methods’ in calculus of variations to minimization problems for integral functionals of the form

$$\int_{\Omega} f(d\omega),$$

where $1 \leq k \leq n$ are integers, $f : \Lambda^k \rightarrow \mathbb{R}$ is a continuous function and ω is a differential $k - 1$ -form on Ω . Before we begin, it will be helpful to take a moment to understand exactly what we are trying to accomplish. The framework for direct methods in classical vectorial calculus of variation concerns itself with minimization problems for functionals of the form

$$\int_{\Omega} f(\nabla \omega),$$

where $N \geq 1$ is an integer, $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ is a continuous function and $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a vector-valued function. The most important convexity condition that ensures the existence of a minimizer, in case of suitable growth assumptions on f is called *quasiconvexity*. The literature for this problem is huge and constitutes the main body of the existing theory (see [25]). However, though attempts to generalize this results to differential operators more general than the gradient has met with some success, the resulting theory is in no way as complete and comprehensive as for the case of the gradient. Such generalizations stems from the observation that $\text{curl}(\nabla u) = 0$. The basic idea is to study minimization of functionals of the form

$$\int_{\Omega} f(\phi), \quad \text{with the constraint } A\phi = 0 \text{ in } \Omega,$$

where A is a first-order differential operator. In the terminology of calculus of variations, the crucial convexity notions in this case is called *A-quasiconvexity* (see [22] and [23], also [29]). In the case of the gradient the operator A is just the curl and in the case we are interested in, A is just the exterior derivative d , by virtue of the identity $dd = 0$. But only a few theorems in the gradient case has an analogue in this general case. If we suppose that the operator A has a

special structure, i.e there exist another first order differential operator B such that $ABv \equiv 0$, i.e the range of B is contained in the kernel of A , the corresponding important convexity notion is called $A - B$ -*quasiconvexity* (see [22], [23], [26]). Clearly, in the classical gradient case, this operator B is the gradient operator and in our case, it is the operator d . However, even with this stronger assumption the situation is not much better. Though it is possible to prove the analogues of a few more results (see e.g Murat[54]), but both the settings are still too general for obtaining a complete characterization theorem of either A -*quasiaffine functions* or $A - B$ -*quasiaffine*, which is crucial for generalizing another extremely important related convexity notion, called *polyconvexity* in the case of the gradient. So our goal is precisely to show that it is possible to develop an analogous, comprehensive theory if we restrict our attention to the operator d or δ .¹

Also we can expect that the theory will have new features due to the special algebraic structure of the exterior product, which are absent in the vectorial calculus of variations, where the relevant algebraic structure is that of the tensor product. It is also possible to obtain a precise relationship between the notions of convexity introduced in this case, namely *ext. polyconvexity*, *ext. quasiconvexity* and *ext. one convexity* with the classical notions of *polyconvexity*, *quasiconvexity* and *rank one convexity* respectively.

The rest of the chapter is organized as follows. We begin with section 3.2, where we define the appropriate convexity notions and derive a few of their properties. In section 3.3, we prove the characterization theorem for *ext. quasiaffine functions*. Section 3.4 explores the relations between these convexity notions in detail both for general functionals and the important special case of quadratic functionals. Section 3.5 deals the question of the precise relationship between these convexity notions and the classical ones. Finally, the chapter ends with section 3.4, where semicontinuity issues are discussed and the existence theorem for minimization problems with *ext. quasiconvex functionals* are obtained. The crucial point for these existence theorems are that growth assumptions on functionals yields only a bound for the L^p norm of $d\omega$, but not for $\nabla\omega$. However, this can be circumvented when the functional depend on $d\omega$, but not explicitly on ω .

3.2 Notions of Convexity

3.2.1 Definitions

We start with the different notions of convexity and affinity.

Definition 3.1 Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) We say that f is *ext. one convex*, if the function

$$g : t \rightarrow g(t) = f(\xi + t\alpha \wedge \beta)$$

¹Some other and related attempts to generalize quasiconvexity, e.g in the setting of elliptic complexes, Carnot groups etc have been tried before (cf. [32], [61]).

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k-1}$. If the function g is affine we say that f is ext. one affine.

(ii) A Borel measurable and locally bounded function f is said to be ext. quasiconvex, if

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{meas}(\Omega)$$

for every bounded open set Ω , for every $\xi \in \Lambda^k$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$. If equality holds, we say that f is ext. quasilinear.

(iii) We say that f is ext. polyconvex, if there exists a convex function

$$F : \Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k} \rightarrow \mathbb{R}$$

such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]})$$

If F is affine, we say that f is ext. polyaffine.

Remark 3.2 (i) The ext. stands for exterior product in the first and third ones and for the exterior derivative for the second one.

(ii) When k is odd (since then $\xi^s = 0$ for every $s \geq 2$) or when $2k > n$ (in particular when $k = n$ or $k = n-1$), then ext. polyconvexity is equivalent to ordinary convexity (see Proposition 3.16).

(iii) When $k = 1$, all the above notions are equivalent to the classical notion of convexity (cf. Theorem 3.12).

(iv) As in Proposition 5.11 of [25], it can easily be shown that if the inequality of ext. quasiconvexity holds for a given bounded open set Ω , it holds for any bounded open sets.

(v) The definition of ext. quasiconvexity is equivalent (as in Proposition 5.13 of [25]) to the following. Let $D = (0,1)^n$, the inequality

$$\int_D f(\xi + d\omega) \geq f(\xi)$$

holds for every $\xi \in \Lambda^k$ and for every

$$\omega \in W_{per}^{1,\infty}(D; \Lambda^{k-1}) = \left\{ \omega \in W^{1,\infty}(D; \Lambda^{k-1}) : \omega \text{ } D\text{-periodic} \right\}.$$

We now present the corresponding definitions when d is replaced by δ .

Definition 3.3 Let $0 \leq k \leq n-1$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) We say that f is int. one convex, if the function

$$g : t \rightarrow g(t) = f(\xi + t\alpha \lrcorner \beta)$$

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k+1}$. If the function g is affine we say that f is int. one affine.

(ii) A Borel measurable and locally bounded function f is said to be int. quasiconvex, if

$$\int_{\Omega} f(\xi + \delta\omega) \geq f(\xi) \text{ meas}(\Omega)$$

for every bounded open set Ω , for every $\xi \in \Lambda^k$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k+1})$. If equality holds, we say that f is int. quasilinear.

(iii) We say that f is int. polyconvex, if there exists a convex function

$$F : \Lambda^{n-k} \times \Lambda^{2(n-k)} \times \dots \times \Lambda^{[n/(n-k)](n-k)} \rightarrow \mathbb{R}$$

such that

$$f(\xi) = F\left(*\xi, (*\xi)^2, \dots, (*\xi)^{[n/(n-k)]}\right).$$

If F is affine, we say that f is int. polyaffine.

There is a natural correspondence between the two sets of definitions, as highlighted in theorem 3.5. To state the theorem, we first need another definition.

Definition 3.4 Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. The Hodge transform of f is the function $f_* : \Lambda^{n-k}(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined as,

$$f_*(\omega) = f(*\omega), \quad \text{for all } \omega \in \Lambda^{n-k}(\mathbb{R}^n)$$

Theorem 3.5 Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. Then,

- (i) f is ext. one convex if and only if f_* is int. one convex.
- (ii) f is ext. quasiconvex if and only if f_* is int. quasiconvex.
- (iii) f is ext. polyconvex if and only if f_* is int. polyconvex.
- (iv) f is convex if and only if f_* is convex.

Proof

(i) f is ext. one convex if and only if

$$g : t \rightarrow g(t) = f(\xi + t\alpha \wedge \beta)$$

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k-1}$. Also, f_* is int. one convex if and only if

$$\bar{g} : t \rightarrow \bar{g}(t) = f_*(\xi + t\alpha \lrcorner \beta)$$

is convex for every $\xi \in \Lambda^{n-k}$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{n-k+1}$. But,

$$\bar{g}(t) = f_*(\xi + t\alpha \lrcorner \beta) = f(*(\xi + t\alpha \lrcorner \beta)) = f(*\xi + t\alpha \wedge *\beta)$$

and conversely,

$$\begin{aligned}
g(t) &= f(\xi + t\alpha \wedge \beta) \\
&= f\left(*\left((-1)^{n(k-1)} * \xi + t\alpha \lrcorner (-1)^{n(k-1)} * \beta\right)\right) \\
&= f_*\left(*(-1)^{n(k-1)} \xi + t\alpha \lrcorner (-1)^{n(k-1)} * \beta\right)
\end{aligned}$$

The result follows.

(ii) This follows from the fact that,

$$\int_{\Omega} f_*(\xi + \delta\omega) = \int_{\Omega} f(*\xi + *\delta\omega) = \int_{\Omega} f\left(*\xi + d((-1)^{n(k-1)} * \omega)\right)$$

and conversely,

$$\int_{\Omega} f(\xi + d\omega) = \int_{\Omega} f\left(*((-1)^{n(k-1)} * \xi + *\delta((-1)^{n(k-1)} * \omega))\right).$$

The result follows.

(iii) Immediate from the definitions.

(iv) Obvious.

This completes the proof. ■

3.2.2 Preliminary lemmas

In this subsection, we state two approximation lemmas which will be used in sequel. We start with the scalar version of the approximation lemma. For the proof, see Lemma 3.10 of [25].

Lemma 3.6 (Scalar approximation lemma) *Let $n \in \mathbb{N}$, $a < b$, $\alpha, \beta \in \mathbb{R}^n$, $t \in [0, 1]$ and let $u_{\alpha, \beta} : \mathbb{R} \rightarrow \mathbb{R}^n$ be defined as*

$$u_{\alpha, \beta}(x) := (t\alpha + (1-t)\beta)x, \text{ for all } x \in \mathbb{R}.$$

Then, for every $\epsilon > 0$, there exist $u \in \text{Aff}_{\text{piece}}([a, b]; \mathbb{R}^n)$ and disjoint open sets $I_{\alpha}, I_{\beta} \subset (a, b)$ such that

1. $\text{meas}(I_{\alpha}) = t(b-a)$ and $\text{meas}(I_{\beta}) = (1-t)(b-a)$,
2. $u(a) = u_{\alpha, \beta}(a)$ and $u(b) = u_{\alpha, \beta}(b)$,
3. $\|u - u_{\alpha, \beta}\|_{L^{\infty}([a, b])} \leq \epsilon$, and
4. $u'(x) = \begin{cases} \alpha, & \text{if } x \in I_{\alpha}, \\ \beta, & \text{if } x \in I_{\beta}. \end{cases}$

We now extend Lemma 3.6 for differential k -forms. See Lemma 3.11 of [25] for the case of the gradient.

Lemma 3.7 *Let $1 \leq k \leq n$, $t \in [0, 1]$ and let $\alpha, \beta \in \Lambda^k$ be such that $\alpha \neq \beta$ and $\alpha - \beta$ is ext.one-divisible. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $\omega : \overline{\Omega} \rightarrow \Lambda^{k-1}$ satisfy*

$$d\omega = t\alpha + (1-t)\beta, \text{ in } \overline{\Omega}.$$

Then, for every $\epsilon > 0$, there exist $\omega_\epsilon \in C_{piece}^1(\overline{\Omega}; \Lambda^{k-1})$ and disjoint open sets $\Omega_\alpha, \Omega_\beta \subset \Omega$ such that

1. $|\text{meas}(\Omega_\alpha) - t \text{meas}(\Omega)| \leq \epsilon$ and $|\text{meas}(\Omega_\beta) - (1-t) \text{meas}(\Omega)| \leq \epsilon$,
2. $\omega_\epsilon = \omega$, in a neighbourhood of $\partial\Omega$,
3. $\|\omega_\epsilon - \omega\|_{L^\infty(\Omega)} \leq \epsilon$,
4. $d\omega_\epsilon(x) = \begin{cases} \alpha, & \text{if } x \in \Omega_\alpha, \\ \beta, & \text{if } x \in \Omega_\beta, \end{cases}$
5. $\text{dist}(d\omega_\epsilon(x); \{t\alpha + (1-t)\beta : t \in [0, 1]\}) \leq \epsilon$, for all $x \in \Omega$ a.e.

Proof Let $\epsilon > 0$ be given. We recall that a k form α is said to be one-divisible, ext.one-divisible or one-decomposable, if there exist $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$ such that $\alpha = a \wedge b$. Now since $\alpha - \beta$ is 1-divisible, there exists $\overline{\omega} \in \Lambda^{k-1} \setminus \{0\}$ and $\nu \in \Lambda^1$, $\|\nu\| = 1$ such that

$$\alpha - \beta = \nu \wedge \overline{\omega}. \quad (3.1)$$

We now consider two cases. In the first case, we assume that

Case 1. $\nu = e^1$.

Note that, by writing Ω as the union of cubes parallel to co-ordinate axes and a set of small positive measure and by setting $\omega_\epsilon = \omega$ on the set of small measure, we may assume that $\Omega = (0, 1)^n$.

Let $\Omega_\epsilon \subset \subset \Omega$, let $\eta \in \text{Aff}_{piece}(\overline{\Omega})$ and let $L > 0$ be such that

$$\text{meas}(\Omega - \Omega_\epsilon) \leq \epsilon \text{ and } \text{supp } \eta \subset \Omega, \quad (3.2)$$

$$0 \leq \eta(x) \leq 1, \text{ for all } x \in \overline{\Omega}, \quad (3.3)$$

$$\eta(x) = 1, \text{ for all } x \in \Omega_\epsilon, \text{ and} \quad (3.4)$$

$$\|D\eta(x)\| \leq \frac{L}{\epsilon}, \text{ for all } x \in \Omega \setminus \Omega_\epsilon \text{ a.e.} \quad (3.5)$$

We invoke Lemma 3.6 at this point. Let us choose δ ,

$$0 < \delta < \min \left\{ \epsilon, \frac{\epsilon^2}{L} \right\}. \quad (3.6)$$

With this δ , using Lemma 3.6, we find $u \in \text{Aff}_{piece}([0, 1]; \Lambda^{k-1})$, where $\Lambda^{k-1} = \Lambda^{k-1}(\mathbb{R}^n)$, and two disjoint open sets $\Omega_1, \Omega_2 \subset (0, 1)$ such that

$$\overline{\Omega_1} \cup \overline{\Omega_2} = [0, 1],$$

$$\text{meas}(\Omega_1) = t \text{ and } \text{meas}(\Omega_2) = (1 - t),$$

$$u(0) = u(1) = 0,$$

$$\|u\|_{L^\infty([0,1])} \leq \delta \text{ and} \quad (3.7)$$

$$u'(s) = \begin{cases} (1-t)\bar{\omega}, & \text{if } s \in \Omega_1, \\ -t\bar{\omega}, & \text{if } s \in \Omega_2. \end{cases} \quad (3.8)$$

Note that, we have applied Lemma 3.6 by setting

$$\alpha = (1-t)\bar{\omega} \text{ and } \beta = -t\bar{\omega},$$

in Lemma 3.6.

We now define $\psi : [0, 1] \times \mathbb{R}^{n-1} \rightarrow \Lambda^{k-1}$ by

$$\psi(x) = \psi(x_1, \dots, x_n) := u(x_1), \text{ for all } x \in [0, 1] \times \mathbb{R}^{n-1}.$$

Therefore,

$$d\psi(x) = e^1 \wedge u'(x_1), \text{ for all } x \in [0, 1] \times \mathbb{R}^{n-1} \text{ a.e.}$$

Indeed, writing

$$u(s) := \sum_{I \in \mathcal{T}^{k-1}} a_I(s) dx^I, \text{ for all } s \in [0, 1],$$

we have

$$\psi(x) = u(x_1) = \sum_{I \in \mathcal{T}^{k-1}} a_I(x_1) dx^I, \text{ for all } x \in [0, 1] \times \mathbb{R}^{n-1}.$$

Hence,

$$\begin{aligned} d\psi(x) &= \sum_{I \in \mathcal{T}^{k-1}} \frac{\partial a_I}{\partial x_1}(x_1) dx^1 \wedge dx^I \\ &= dx^1 \wedge \left(\sum_{I \in \mathcal{T}^{k-1}} a'_I(x_1) dx^I \right) = e^1 \wedge u'(x_1), \text{ for all } x \in [0, 1] \times \mathbb{R}^{n-1} \text{ a.e.} \end{aligned}$$

Therefore, it follows from Equations (3.1) and (3.8) that

$$d\psi(x) = e^1 \wedge u'(x_1) = \begin{cases} (1-t)(\alpha - \beta), & \text{if } x \in \Omega_1 \times (0, 1)^{n-1}, \\ -t(\alpha - \beta), & \text{if } x \in \Omega_2 \times (0, 1)^{n-1}. \end{cases} \quad (3.9)$$

Therefore,

$$d\psi + d\omega \in \{\alpha, \beta\}, \text{ a.e. in } \Omega. \quad (3.10)$$

Finally, we define $\omega_\epsilon : \bar{\Omega} = [0, 1]^n \rightarrow \Lambda^{k-1}$ by

$$\omega_\epsilon(x) := \eta(x)(\psi(x) + \omega(x)) + (1 - \eta(x))\omega(x), \text{ for all } x \in \bar{\Omega}.$$

We now show that ω_ϵ satisfies the conclusions of the lemma with

$$\Omega_\alpha := \{x \in \Omega_\epsilon : x_1 \in \Omega_1\} \text{ and } \Omega_\beta := \{x \in \Omega_\epsilon : x_1 \in \Omega_2\}.$$

Indeed, in a neighbourhood of $\partial\Omega$, it follows from (3.2) that $\omega_\epsilon = \omega$. Furthermore, using Equations (3.3), (3.7) and (3.6), we deduce that

$$\|\omega_\epsilon - \omega\|_{L^\infty(\Omega)} = \|\eta\psi\|_{L^\infty(\Omega)} \leq \|\psi\|_{L^\infty(\Omega)} \leq \delta < \epsilon.$$

We now calculate $d\omega_\epsilon$. To show this, we note that

$$d\omega_\epsilon = d\eta \wedge \psi + \eta d\psi + d\omega, \text{ a.e. in } \Omega.$$

Using Equations (3.4) and (3.9), we find that

$$d\omega_\epsilon = d\psi + d\omega = \begin{cases} \alpha, & \text{in } \Omega_\alpha, \\ \beta, & \text{in } \Omega_\beta. \end{cases} \quad (3.11)$$

It remains to prove that

$$\text{dist}(d\omega_\epsilon; \text{co}\{\alpha, \beta\}) \leq \epsilon, \text{ a.e. in } \Omega. \quad (3.12)$$

Since

$$d\omega = t\alpha + (1-t)\beta \in \text{co}\{\alpha, \beta\}, \text{ in } \Omega,$$

it follows from Equation (3.10) that

$$\eta d\psi + d\omega = \eta(d\psi + d\omega) + (1-\eta)d\omega \in \text{co}\{\alpha, \beta\}, \text{ in } \Omega.$$

Furthermore, using Equations (3.5) and (3.6), it is easy to check that

$$\|d\eta \wedge \psi\|_{L^\infty(\bar{\Omega})} \leq \|d\eta\|_{L^\infty(\bar{\Omega})} \|\psi\|_{L^\infty(\bar{\Omega})} \leq \frac{L}{\epsilon} \delta \leq \epsilon,$$

which proves Equation (3.12). This proves the theorem for the first case. We now consider the general case.

Case 2. General ν .

Let $T \in O(n)$ be such that $T^t(\nu) = e^1$. Let us define

$$\Omega^* := T^t(\Omega), \alpha^* := T^*\alpha, \text{ and } \beta^* := T^*\beta,$$

where T^* is the pullback of T . Note that,

$$\begin{aligned} \alpha^* - \beta^* &= T^*(\alpha - \beta) = T^*(\nu \wedge \bar{\omega}) = T^*\nu \wedge T^*\bar{\omega} \\ &= T^t(\nu) \wedge T^*\bar{\omega} = e^1 \wedge T^*\bar{\omega}. \end{aligned}$$

Using Case 1, we find $\omega^* \in C_{\text{piece}}^1(\bar{\Omega}^*; \Lambda^{k-1})$ and disjoint open sets $\Omega_{\alpha^*}^*, \Omega_{\beta^*}^* \subset \Omega^*$ such that

$$1. \quad |\text{meas}(\Omega_{\alpha^*}^*) - t \text{meas}(\Omega^*)| \leq \epsilon \text{ and } |\text{meas}(\Omega_{\beta^*}^*) - (1-t) \text{meas}(\Omega^*)| \leq \epsilon,$$

2. $\omega_\epsilon^* = \omega^*$, in a neighbourhood of $\partial\Omega^*$, where ω^* satisfies

$$d\omega^* = t\alpha^* + (1-t)\beta^*, \text{ in } \overline{\Omega}.$$

3. $\|\omega_\epsilon^* - \omega^*\|_{L^\infty(\overline{\Omega^*})} \leq \epsilon$,

$$4. \quad d\omega_\epsilon^*(x) = \begin{cases} \alpha^*, & \text{if } x \in \Omega_{\alpha^*}^*, \\ \beta^*, & \text{if } x \in \Omega_{\beta^*}^*, \text{ and} \end{cases}$$

5. $\text{dist}(d\omega_\epsilon^*(x); \text{co}\{\alpha^*, \beta^*\}) \leq \epsilon$, for all $x \in \Omega^*$ a.e.

Then, it is easy to check that the function $\omega_\epsilon \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \Lambda^{k-1})$ defined as

$$\omega_\epsilon(x) := ((T^t)^* \omega_\epsilon^*)(x), \text{ for all } x \in \overline{\Omega},$$

satisfies all the desired properties. To prove this, it is enough to observe that

$$d\omega_\epsilon = (T^t)^* d\omega^*, \text{ a.e. in } \Omega.$$

This proves the theorem. ■

Now we present an interesting observation which we will not need, but nonetheless we prove it here in full. See Ball-James [7] for the case of the gradient.

Proposition 3.8 *Let $0 \leq k \leq n-1$, $\alpha, \beta \in \Lambda^{k+1}$ and $\Omega \subset \mathbb{R}^n$ is open, bounded, smooth and contractible. Then there exists $\omega \in W^{1,\infty}(\Omega; \Lambda^k)$ satisfying*

$$d\omega \in \{\alpha, \beta\} \text{ a.e. in } \Omega,$$

taking both values, if and only if $a \wedge (\alpha - \beta) = 0$ for some $a \in \Lambda^1$.

Proof (\Rightarrow) Define

$$\Omega_\alpha := \{x \in \Omega : d\omega(x) = \alpha\} \quad \text{and} \quad \Omega_\beta := \{x \in \Omega : d\omega(x) = \beta\}.$$

Also set

$$\phi(x) = \omega(x) - \frac{1}{k+1}(x \lrcorner \beta) \quad \text{for every } x \in \Omega.$$

Note that $\phi \in W^{1,\infty}(\Omega; \Lambda^k)$ and

$$d\phi = \begin{cases} \alpha - \beta & \text{in } \Omega_\alpha, \\ 0 & \text{in } \Omega_\beta. \end{cases}$$

Thus $d\phi(x) = \chi_{\Omega_\alpha}(x)(\alpha - \beta)$. Since $\chi_{\Omega_\alpha}(x)$ is not constant, there exists $\rho \in C_c^\infty(\Omega)$ such that,

$$a := \int_\Omega \chi_{\Omega_\alpha}(x) \nabla \rho(x) \neq 0.$$

Clearly we can also assume $|a| = 1$.

Now we claim that $a \wedge (\alpha - \beta) = 0$. Indeed, we have,

$$a \wedge (\alpha - \beta) = \int_{\Omega} \chi_{\Omega_a}(x) \nabla \rho(x) \wedge (\alpha - \beta) = \int_{\Omega} \nabla \rho(x) \wedge d\phi(x) = - \int_{\Omega} d(\nabla \rho(x) \wedge \phi(x)).$$

Since $\nabla \rho(x) \wedge \phi(x) \in W_0^{1,\infty}(\Omega; \Lambda^{k+1})$, we obtain the claim by integration by parts .

(\Leftarrow) Conversely, suppose $a \wedge (\alpha - \beta) = 0$ for some $a \in \Lambda^1$. Then there exists $b \in \Lambda^k$ such that $\alpha - \beta = a \wedge b$. Now we find $u \in W^{1,\infty}(\Omega)$ such that

$$\nabla u \in \{a, 0\} \quad \text{a.e. in } \Omega,$$

taking both values. Now we define

$$\omega(x) = u(x)b + \frac{1}{k+1}(x \lrcorner \beta) \quad \text{for every } x \in \Omega.$$

Then $\omega \in W^{1,\infty}(\Omega; \Lambda^k)$ and we have, for a.e. $x \in \Omega$,

$$d\omega = \nabla u \wedge b + \beta = \{a \wedge b, 0\} + \beta = \{\alpha, \beta\}.$$

This finishes the proof. ■

Remark 3.9 *The natural question that what we can prove if $d\omega$ takes s distinct values a.e. for small $s > 2$ would be an interesting question worth looking into. In the classical case, this is addressed by Šverák [62], [64] and Zhang [75] for the case $s = 3$, Chlebik-Kirchheim [19] for $s = 4$ and Kirchheim-Presiss [41] for $s \geq 5$.*

3.2.3 Main properties

The different notions of convexity are related as follows.

Theorem 3.10 *Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. Then*

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex} \Rightarrow f \text{ ext. one convex}.$$

Moreover if $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ is ext. one convex, then f is locally Lipschitz. If, in addition f is C^2 , then for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k-1}$,

$$\sum_{I, J \in \mathcal{T}^k} \frac{\partial^2 f(\xi)}{\partial \xi_I \partial \xi_J} (\alpha \wedge \beta)_I (\alpha \wedge \beta)_J \geq 0.$$

Remark 3.11 (i) *As already pointed out, when k is odd or when $2k > n$ (in particular when $k = n$ or $k = n - 1$), then ext. polyconvexity is equivalent to the classical convexity.*

(ii) *Since ext. one convex functions are locally Lipschitz continuous so are ext. one quasiconvex or ext. one polyconvex functions.*

Proof (i) In view of Theorem 3.54, the result follows at once from the general fact (see Theorem 5.3 in [25])

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

However, we will also provide a direct proof of these facts here.

Step 1. The implication

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex}$$

is trivial.

Step 2. The statement

$$f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex}$$

is proved as follows. Observe first that if $\xi \in \Lambda^k$ and $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$, then

$$\int_{\Omega} (\xi + d\omega)^s = \xi^s \text{ meas}(\Omega), \quad \text{for every integer } s. \quad (3.13)$$

We proceed by induction on s . The case $s = 1$ is trivial, so we assume that the result has already been established for $s - 1$ and we prove it for s . Note that

$$\begin{aligned} (\xi + d\omega)^s &= \xi \wedge (\xi + d\omega)^{s-1} + d\omega \wedge (\xi + d\omega)^{s-1} \\ &= \xi \wedge (\xi + d\omega)^{s-1} + d \left[\omega \wedge (\xi + d\omega)^{s-1} \right]. \end{aligned}$$

Integrating, using induction for the first integral on the right hand side and the fact that $\omega = 0$ on $\partial\Omega$ for the second one, we have indeed shown (3.13). We can now conclude. Since f is ext. polyconvex, we can find a convex function

$$F : \Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k} \rightarrow \mathbb{R}$$

such that

$$f(\xi) = F(\xi, \xi^2, \dots, \xi^{[n/k]}).$$

Using Jensen inequality we find,

$$\frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\omega) \geq F\left(\frac{1}{\text{meas } \Omega} \int_{\Omega} (\xi + d\omega), \dots, \frac{1}{\text{meas } \Omega} \int_{\Omega} (\xi + d\omega)^{[n/k]}\right).$$

Invoking (3.13), we have indeed obtained that

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{ meas } \Omega,$$

and the proof of Step 2 is complete.

Step 3. Let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext. quasiconvex and let $\xi \in \Lambda^k$, $a \in \Lambda^1$, $b \in \Lambda^{k-1}$ be fixed. We need to show that the function

$$g : t \rightarrow g(t) = f(\xi + ta \wedge b)$$

is convex. To show this, let $\lambda \in [0, 1]$, $t, s \in \mathbb{R}$. We shall show,

$$g(\lambda t + (1 - \lambda)s) \leq \lambda g(t) + (1 - \lambda)g(s).$$

But this is equivalent to showing that

$$f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b) \leq \lambda f(\xi + ta \wedge b) + (1 - \lambda)f(\xi + sa \wedge b).$$

We assume $t \neq s$, as otherwise the inequality is trivial.

Using Lemma 3.7, we find disjoint open sets $\Omega_1, \Omega_2 \subset \Omega$ and $\phi \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$ such that

1. $|\text{meas}(\Omega_1) - \lambda \text{meas}(\Omega)| \leq \epsilon$ and $|\text{meas}(\Omega_2) - (1 - \lambda) \text{meas}(\Omega)| \leq \epsilon$,
2. $\|\phi\|_{L^\infty(\Omega)} < \infty$,
3. $d\phi(x) = \begin{cases} (1 - \lambda)(t - s)a \wedge b, & \text{if } x \in \Omega_1, \\ -\lambda(t - s)a \wedge b, & \text{if } x \in \Omega_2. \end{cases}$

Since f is ext. quasiconvex, we have,

$$\begin{aligned} & \int_{\Omega} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + d\phi) \\ &= \int_{\Omega_1} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + (1 - \lambda)(t - s)a \wedge b) \\ & \quad + \int_{\Omega_2} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + \lambda(t - s)a \wedge b) \\ & \quad + \int_{\Omega \setminus (\Omega_1 \cup \Omega_2)} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + d\phi) \\ &= \text{meas}(\Omega_1)f(\xi + ta \wedge b) + \text{meas}(\Omega_2)f(\xi + sa \wedge b) \\ & \quad + \int_{\Omega \setminus (\Omega_1 \cup \Omega_2)} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + d\phi). \end{aligned}$$

But we have,

$$\text{meas}(\Omega \setminus (\Omega_1 \cup \Omega_2)) = \lambda \text{meas}(\Omega) - \text{meas}(\Omega_1) + (1 - \lambda) \text{meas}(\Omega) - \text{meas}(\Omega_2) \leq 2\epsilon.$$

Also, we have,

$$\begin{aligned} \text{meas}(\Omega_1)f(\xi + ta \wedge b) &\leq \lambda \text{meas}(\Omega)f(\xi + ta \wedge b) + \epsilon f(\xi + ta \wedge b), & \text{if } f(\xi + ta \wedge b) \geq 0, \\ \text{meas}(\Omega_1)f(\xi + ta \wedge b) &\leq \text{meas}(\Omega)f(\xi + ta \wedge b) - \epsilon f(\xi + ta \wedge b), & \text{if } f(\xi + ta \wedge b) < 0. \end{aligned}$$

Similar inequalities hold for $\text{meas}(\Omega_2)f(\xi + sa \wedge b)$. Combining them and letting $\epsilon \rightarrow 0$, we obtain,

$$\begin{aligned} \text{meas}(\Omega)f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b) &\leq \int_{\Omega} f(\xi + (\lambda t + (1 - \lambda)s)a \wedge b + d\phi) \\ &\quad \lambda \text{meas}(\Omega)f(\xi + ta \wedge b) + (1 - \lambda) \text{meas}(\Omega)f(\xi + sa \wedge b). \end{aligned}$$

This proves the result.

(ii) The fact that f is locally Lipschitz follows from the observation that any ext. one convex function is in fact separately convex. These last functions are known to be locally Lipschitz (cf. Theorem 2.31 in [25]).

(iii) We next assume that f is C^2 . By definition the function

$$g : t \rightarrow g(t) = f(\xi + t\alpha \wedge \beta)$$

is convex for every $\xi \in \Lambda^k$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k-1}$. Since f is C^2 , we get the claim from the fact that $g''(0) \geq 0$. ■

There are some cases where all the different notions are equivalent.

Theorem 3.12 *Let $k = 1, n - 1, n$ or $k = n - 2$ and n odd and let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. Then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex}.$$

Remark 3.13 *The last result, i.e. when $k = n - 2$, is false when n is even, as the following simple example shows. Let $f : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$ be defined by*

$$f(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle.$$

The function f is clearly ext. polyconvex but not convex. However as soon as $n \geq 5$ and $k = n - 2$ (since then $2k > n$), then, as already mentioned, convexity and ext. polyconvexity are equivalent. However this is not the case with ext. quasiconvexity (see Theorem 3.30 (iii)).

Proof In all cases under consideration any $\xi \in \Lambda^k$ can be written as (see Remark 2.11)

$$\xi = \alpha \wedge \beta$$

with $\alpha \in \Lambda^1$ and $\beta \in \Lambda^{k-1}$. The result then follows at once. ■

We now give an equivalent formulation of ext. quasiconvexity.

Proposition 3.14 *Let $f : \Lambda^k \rightarrow \mathbb{R}$ be continuous, $1 < p < \infty$, $c > 0$ be such that, for every $\xi \in \Lambda^k$,*

$$|f(\xi)| \leq c(1 + |\xi|^p).$$

The following two statements are then equivalent.

(i) The function f verifies

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{ meas } \Omega$$

for every bounded smooth open set $\Omega \subset \mathbb{R}^n$, for every $\xi \in \Lambda^k$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$.

(ii) For every bounded smooth open set $\Omega \subset \mathbb{R}^n$, for every $\xi \in \Lambda^k$ and for every $\omega \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1})$

$$\int_{\Omega} f(\xi + d\omega) \geq f(\xi) \text{ meas } \Omega.$$

Remark 3.15 Given a function $f : \Lambda^k \rightarrow \mathbb{R}$ the ext. quasiconvex envelope, which is the largest ext quasiconvex function below f , is given by (as in Theorem 6.9 of [25])

$$\begin{aligned} Q_{\text{ext}} f(\xi) &= \inf \left\{ \frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\omega) : \omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1}) \right\} \\ &= \inf \left\{ \frac{1}{\text{meas } \Omega} \int_{\Omega} f(\xi + d\omega) : \omega \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1}) \right\}. \end{aligned}$$

Proof (i) \Rightarrow (ii). Let $\psi \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$. Appealing to Theorem 2.43, we can find $\omega \in W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\begin{cases} d\omega = d\psi & \text{in } \Omega \\ \delta\omega = 0 & \text{in } \Omega \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The result follows by approximating ω by $W_{\delta,T}^{1,\infty}$ forms, using the bound on the function f .

(ii) \Rightarrow (i). Let $\psi \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^{k-1})$. Then, by Theorem 2.47, we can find $\omega \in W_0^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\begin{cases} d\omega = d\psi & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The result follows by approximating ω by $W_0^{1,\infty}$ forms, using the bound on the function f . ■

We finally have also another formulation of ext. polyconvexity.

Proposition 3.16 Let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) The function f is ext. polyconvex if and only if, for every $\xi \in \Lambda^k$, there exist $c_s = c_s(\xi) \in \Lambda^{ks}$, $1 \leq s \leq [n/k]$, such that

$$f(\eta) \geq f(\xi) + \sum_{s=1}^{[n/k]} \langle c_s(\xi); \eta^s - \xi^s \rangle, \quad \text{for every } \eta \in \Lambda^k.$$

(ii) Let

$$\tau = \dim(\Lambda^k \times \Lambda^{2k} \times \dots \times \Lambda^{[n/k]k}) = \sum_{s=1}^{[n/k]} \binom{n}{ks}.$$

Then the function f is ext. polyconvex if and only if, for any collection $\{t_i, \xi_i\}_{i=1}^{\tau+1} \subset \mathbb{R}^+ \times \Lambda^k$, with $\sum_{i=1}^{\tau+1} t_i = 1$ and $\sum_{i=1}^{\tau+1} t_i \xi^s = \left(\sum_{i=1}^{\tau+1} t_i \xi_i \right)^s$, for every $1 \leq s \leq [n/k]$,

we have ,

$$f\left(\sum_{i=1}^{\tau+1} t_i \xi_i\right) \leq \sum_{i=1}^{\tau+1} t_i f(\xi_i).$$

(iii) If either k is odd or $2k > n$, then ext. polyconvexity is equivalent to ordinary convexity.

Proof (i) (\Rightarrow) Since f is ext polyconvex, there exists a convex function F such that

$$f(\xi) = F\left(\xi, \xi^2, \dots, \xi^{[n/k]}\right).$$

F being convex, there exist, for every $\xi \in \Lambda^k$, $c_s = c_s(\xi) \in \Lambda^{ks}$, $1 \leq s \leq [n/k]$, such that

$$f(\eta) - f(\xi) = F\left(\eta, \dots, \eta^{[n/k]}\right) - F\left(\xi, \dots, \xi^{[n/k]}\right) \geq \sum_{s=1}^{[n/k]} \langle c_s; \eta^s - \xi^s \rangle$$

as claimed.

(\Leftarrow) Conversely fix $\xi \in \Lambda^k$ and let, for $\theta \in \Lambda^k \times \dots \times \Lambda^{[n/k]k}$,

$$F(\theta) = \sup_{\theta \in \Lambda^k \times \dots \times \Lambda^{[n/k]k}} \left\{ f(\xi) + \sum_{s=1}^{[n/k]} \left\langle c_s(\xi); \theta - \left(\xi, \dots, \xi^{[n/k]}\right) \right\rangle \right\}.$$

Clearly F is convex. Then it is easy to see, as in Theorem 5.6 in [25], that

$$f(\xi) = F\left(\xi, \dots, \xi^{[n/k]}\right)$$

and thus f is ext. polyconvex.

(ii) Like (i) above, this is again a consequence of convexity and Carathéodory theorem for convex functions on \mathbb{R}^d and $d+1$ -simplexes. The proof is essentially the same as that of Theorem 5.6 in [25], with the obvious modifications.

(iii) When k is odd, then $\xi^s = 0$ for every $s \geq 2$ and similarly when $2k > n$. The result follows at once from this observation. ■

3.3 The quasilinear case

3.3.1 Some preliminary results

We start with two elementary results.

Lemma 3.17 Let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext. one affine with $1 \leq k \leq n$. Then

$$f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) = f(\xi) + \sum_{i=1}^N t_i [f(\xi + \alpha_i \wedge a) - f(\xi)]$$

for every $t_i \in \mathbb{R}$, $\xi \in \Lambda^k$, $\alpha_i \in \Lambda^{k-1}$, $a \in \Lambda^1$.

Proof Step 1. It is easy to see that f is C^1 (in fact C^∞). We therefore find

$$\begin{aligned} f(\xi + t\alpha \wedge a) &= f(\xi) + t \langle \nabla f(\xi); \alpha \wedge a \rangle \\ f(\xi + \alpha \wedge a) &= f(\xi) + \langle \nabla f(\xi); \alpha \wedge a \rangle \end{aligned}$$

and thus

$$f(\xi + t\alpha \wedge a) = f(\xi) + t[f(\xi + \alpha \wedge a) - f(\xi)].$$

Step 2. Let us first prove that

$$f(\xi + \alpha \wedge a + \beta \wedge a) + f(\xi) = f(\xi + \alpha \wedge a) + f(\xi + \beta \wedge a).$$

First assume that $s \neq 0$. We have, using Step 1, that

$$\begin{aligned} &f(\xi + s\alpha \wedge a + \beta \wedge a) \\ &= f\left(\xi + s\left(\alpha + \frac{1}{s}\beta\right) \wedge a\right) \\ &= f(\xi) + s\left[f\left(\xi + \left(\alpha + \frac{1}{s}\beta\right) \wedge a\right) - f(\xi)\right] \end{aligned}$$

and hence, using Step 1 again,

$$\begin{aligned} &f(\xi + s\alpha \wedge a + \beta \wedge a) \\ &= f(\xi) + s\left\{f(\xi + \alpha \wedge a) + \frac{1}{s}[f(\xi + \alpha \wedge a + \beta \wedge a) - f(\xi + \alpha \wedge a)] - f(\xi)\right\} \\ &= f(\xi) + s[f(\xi + \alpha \wedge a) - f(\xi)] + [f(\xi + \alpha \wedge a + \beta \wedge a) - f(\xi + \alpha \wedge a)]. \end{aligned}$$

Since f is continuous, we have the result by letting $s \rightarrow 0$.

Step 3. We now prove the claim. We proceed by induction. The case $N = 1$ is just Step 1. We first use the induction hypothesis to write

$$\begin{aligned} &f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) \\ &= f\left(\xi + t_N \alpha_N \wedge a + \sum_{i=1}^{N-1} t_i \alpha_i \wedge a\right) \\ &= f(\xi + t_N \alpha_N \wedge a) + \sum_{i=1}^{N-1} t_i [f(\xi + t_N \alpha_N \wedge a + \alpha_i \wedge a) - f(\xi + t_N \alpha_N \wedge a)]. \end{aligned}$$

We then appeal to Step 1 to get

$$\begin{aligned}
& f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) \\
&= f(\xi) + t_N [f(\xi + \alpha_N \wedge a) - f(\xi)] \\
&+ \sum_{i=1}^{N-1} t_i \left\{ \begin{aligned} & f(\xi + \alpha_i \wedge a) + t_N [f(\xi + \alpha_i \wedge a + \alpha_N \wedge a) - f(\xi + \alpha_i \wedge a)] \\ & - f(\xi) - t_N [f(\xi + \alpha_N \wedge a) - f(\xi)] \end{aligned} \right\}
\end{aligned}$$

and thus

$$\begin{aligned}
f\left(\xi + \sum_{i=1}^N t_i \alpha_i \wedge a\right) &= f(\xi) + \sum_{i=1}^N t_i [f(\xi + \alpha_i \wedge a) - f(\xi)] \\
&+ t_N \sum_{i=1}^{N-1} t_i \left\{ \begin{aligned} & f(\xi + \alpha_i \wedge a + \alpha_N \wedge a) - f(\xi + \alpha_i \wedge a) \\ & - f(\xi + \alpha_N \wedge a) + f(\xi) \end{aligned} \right\}.
\end{aligned}$$

Appealing to Step 2, we see that each term in the last term vanishes and therefore the induction reasoning is complete and this achieves the proof of the lemma. ■

We have as an immediate consequence the following result.

Corollary 3.18 *Let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext. one affine with $1 \leq k \leq n$. Then*

$$\begin{aligned}
& [f(\xi + \alpha \wedge a + \beta \wedge b) - f(\xi)] + [f(\xi + \beta \wedge a + \alpha \wedge b) - f(\xi)] \\
&= [f(\xi + \alpha \wedge a) - f(\xi)] + [f(\xi + \beta \wedge a) - f(\xi)] \\
&+ [f(\xi + \alpha \wedge b) - f(\xi)] + [f(\xi + \beta \wedge b) - f(\xi)].
\end{aligned}$$

for every $\xi \in \Lambda^k$, $\alpha, \beta \in \Lambda^{k-1}$, $a, b \in \Lambda^1$.

Proof Step 1. It follows from Lemma 3.17 that

$$\begin{aligned}
f(\xi + \alpha \wedge a) + f(\xi + \beta \wedge a) &= f(\xi) + f(\xi + (\alpha + \beta) \wedge a) \\
f(\xi + \alpha \wedge b) + f(\xi + \beta \wedge b) &= f(\xi) + f(\xi + (\alpha + \beta) \wedge b)
\end{aligned}$$

and thus

$$\begin{aligned}
& f(\xi + \alpha \wedge a) + f(\xi + \beta \wedge a) + f(\xi + \alpha \wedge b) + f(\xi + \beta \wedge b) \\
&= 2f(\xi) + f(\xi + (\alpha + \beta) \wedge a) + f(\xi + (\alpha + \beta) \wedge b).
\end{aligned}$$

Step 2. Observe that

$$\begin{aligned}
\alpha \wedge a + \beta \wedge b &= (\alpha + \beta) \wedge a + \beta \wedge (b - a) \\
\beta \wedge a + \alpha \wedge b &= (\alpha + \beta) \wedge a + \alpha \wedge (b - a)
\end{aligned}$$

and thus

$$\begin{aligned} & f(\xi + \alpha \wedge a + \beta \wedge b) + f(\xi + \beta \wedge a + \alpha \wedge b) \\ &= f(\xi + (\alpha + \beta) \wedge a + \beta \wedge (b - a)) + f(\xi + (\alpha + \beta) \wedge a + \alpha \wedge (b - a)). \end{aligned}$$

We therefore have from Lemma 3.17 that

$$\begin{aligned} & f(\xi + \alpha \wedge a + \beta \wedge b) + f(\xi + \beta \wedge a + \alpha \wedge b) \\ &= f(\xi + (\alpha + \beta) \wedge a) + f(\xi + (\alpha + \beta) \wedge a + (\alpha + \beta) \wedge (b - a)) \\ &= f(\xi + (\alpha + \beta) \wedge a) + f(\xi + (\alpha + \beta) \wedge b). \end{aligned}$$

Comparing Step 1 with the above identity, we have indeed obtained the claim. ■

We also have another corollary which we will not be needing in the sequel, but we nonetheless present it here in full.

Corollary 3.19

$$\sum_{i=1}^N \sum_{j=1}^N \{f(\omega + a_i \wedge \alpha_j) - f(\omega)\} = f\left(\omega + \left(\sum_{i=1}^N a_i\right) \wedge \left(\sum_{j=1}^N \alpha_j\right)\right) - f(\omega) \quad (3.14)$$

for all $f : \Lambda^k \rightarrow \mathbb{R}$ ext. one affine with $1 \leq k \leq n$ and any $\omega \in \Lambda^k$, $a_i \in \Lambda^1$ for all $1 \leq i \leq N$, $\alpha_j \in \Lambda^{k-1}$ for all $1 \leq j \leq N$ and any $N \geq 1$.

Proof

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N \{f(\omega + a_i \wedge \alpha_j) - f(\omega)\} \\ &= \sum_{i=1}^n f(\omega + a_i \wedge (\sum_{j=1}^N \alpha_j)) - f(\omega) \quad [\text{By Lemma (3.17)}] \\ &= f(\omega + (\sum_{i=1}^N a_i) \wedge (\sum_{j=1}^N \alpha_j)) - f(\omega) \quad [\text{By Lemma (3.17) again}] \end{aligned}$$

■

3.3.2 The characterization theorem

Now we are going to present the characterization theorem for ext. quasilinear functions. The proof given here is, in a way, the cleanest direct proof of this result and is essentially the proof in [10]. Another proof, using the result of classical vectorial calculus of variation can be found in section 3.5, which is the one in [11]. Another direct algebraic proof, which is more constructive but also a bit messy, can be found in [12].

Theorem 3.20 *Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. The following statements are then equivalent.*

(i) f is ext. polyaffine.

(ii) f is ext. quasilinear.

(iii) f is ext. one affine.

(iv) There exist $c_s \in \Lambda^{ks}$, $0 \leq s \leq [n/k]$, such that, for every $\xi \in \Lambda^k$,

$$f(\xi) = \sum_{s=0}^{[n/k]} \langle c_s; \xi^s \rangle.$$

Remark 3.21 (i) $\xi^0 \in \Lambda^0$ is defined to be 1 for any $\xi \in \Lambda^k$.

(ii) When k is odd (since then $\xi^s = 0$ for every $s \geq 2$) or when $2k > n$ (in particular when $k = n$ or $k = n - 1$), then all the statements are equivalent to f affine.

(iii) In the terminology of Ball [4], these functions are precisely the ‘Null Lagrangians’ in this context.

Proof The statements

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

follow at once from Theorem 3.37. The statement

$$(iv) \Rightarrow (i)$$

is a direct consequence of the definition of ext. polyconvexity. So it only remains to prove

$$(iii) \Rightarrow (iv).$$

We divide the proof into two steps.

Step 1. We first show that f is a polynomial of the form

$$f(\xi) = \sum_{r=0}^{[n/k]} f_r(\xi) \quad \text{where} \quad f_r(\xi) = \sum_{I_k^1, \dots, I_k^r} c_{I_k^1, \dots, I_k^r} \xi_{I_k^1} \cdots \xi_{I_k^r} \quad (3.15)$$

with $c_{I_k^1, \dots, I_k^r} \in \mathbb{R}$ and the ordered multiindices

$$I_k^1 = (i_1^1, \dots, i_k^1), \dots, I_k^r = (i_1^r, \dots, i_k^r)$$

have no index in common. Moreover each of the f_r is ext. one affine. Once the above statement will be proved we decide, in order to avoid any ambiguity, to fix the order in which we take the ordered multiindices I_k^1, \dots, I_k^r and we choose that

$$i_1^1 < \dots < i_1^r.$$

The present step will be obtained in the next two substeps.

Step 1.1. We first prove that f must be a polynomial of degree at most n of the form

$$f(\xi) = \sum_{r=0}^n f_r(\xi) \quad (3.16)$$

where the f_s are homogeneous polynomial of degree s and each of them is ext. one affine. So let us show (3.16). We proceed by induction on n . The case $n = 1$ is trivial. We write

$$\xi_N = \sum_{2 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 i_2 \dots i_k} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}$$

so that

$$\xi = \sum_{2 \leq i_2 < \dots < i_k \leq n} \xi_{1 i_2 \dots i_k} e^1 \wedge e^{i_2} \wedge \dots \wedge e^{i_k} + \xi_N.$$

We therefore can invoke Lemma 3.17 to obtain

$$f(\xi) = f(\xi_N) + \sum_{2 \leq i_2 < \dots < i_n \leq n} \xi_{1 i_2 \dots i_n} [f(\xi_N + e^1 \wedge e^{i_2} \wedge \dots \wedge e^{i_n}) - f(\xi_N)].$$

We then apply the hypothesis of induction to

$$f(\xi_N) \quad \text{and} \quad [f(\xi_N + e^1 \wedge e^{i_2} \wedge \dots \wedge e^{i_n}) - f(\xi_N)]$$

to get that both terms are polynomials of degree at most $(n - 1)$. The fact that each of the f_s is ext. one affine is obvious, since the f_s have different degrees of homogeneity.

Step 1.2. Each of the f_s in (3.16) being a homogeneous polynomial of degree s we can write

$$f_s(\xi) = \sum_{I_k^1, \dots, I_k^s} c_{I_k^1 \dots I_k^s} \xi_{I_k^1} \dots \xi_{I_k^s} \quad (3.17)$$

where $c_{I_k^1 \dots I_k^s} \in \mathbb{R}$. We now claim that the ordered multiindices, in (3.17),

$$I_k^1 = (i_1^1, \dots, i_k^1), \dots, I_k^s = (i_1^s, \dots, i_k^s)$$

have no index in common, so in particular we deduce that the polynomial f has a degree at most $[n/k]$. We proceed by contradiction and assume that one of the index appears more than once, say r times, $2 \leq r \leq s$. This means that there exist ordered multiindices J_k^1, \dots, J_k^s so that

$$c_{J_k^1 \dots J_k^s} \neq 0$$

and the J_k^1, \dots, J_k^r have one index in common say, in order not to burden even more the notations (this can be achieved by relabeling), that this index is 1 and that it appears in the first r multiindices J_k^1, \dots, J_k^r so that

$$j_1^1 = \dots = j_1^r = 1.$$

We then choose $t \in \mathbb{R}$ and

$$\xi = t \sum_{a=1}^r e^{J_k^a} + \sum_{a=r+1}^s e^{J_k^a}$$

in order to have

$$\xi_{J_k^1} = \cdots = \xi_{J_k^r} = t \quad \text{and} \quad \xi_{J_k^{r+1}} = \cdots = \xi_{J_k^s} = 1$$

while all the other coefficients $\xi_{I_k^m}$ are 0. We therefore have according to (3.17) that

$$f_s(\xi) = t^r c_{J_k^1 \dots J_k^s}.$$

However, letting $\xi_N = \sum_{a=r+1}^s e^{J_k^a}$, we should have, since f_s is ext. one affine and according to Lemma 3.17, that $f_s(\xi)$ is linear in the variable t , more precisely

$$f_s(\xi) = f_s\left(\xi_N + t \sum_{a=1}^r e^{J_k^a}\right) = f_s(\xi_N) + t \sum_{a=1}^r [f_s(\xi_N + e^{J_k^a}) - f_s(\xi_N)].$$

This is the desired contradiction. The result is therefore established.

Step 2. From now on we assume that f and f_s are as in (3.15). So the theorem will be proved if we can show that

$$f_s(\xi) = \langle c_s; \xi^s \rangle. \quad (3.18)$$

The above statement is equivalent to proving that the $c_{I_k^1 \dots I_k^s}$ defined in (3.15) satisfy

$$c_{\sigma(I_k^1 \dots I_k^s)} = \text{sgn}(\sigma) c_{I_k^1 \dots I_k^s} \quad (3.19)$$

where σ is a permutation of the indices that respect the order defined in Step 1.

Step 2.1. Let us first show that (3.18) is equivalent to (3.19). The fact that (3.18) implies (3.19) is obvious so we need to prove only the reverse implication. We fix a set of s distinct ordered multiindices

$$I_k^1 = (i_1^1, \dots, i_k^1), \dots, I_k^s = (i_1^s, \dots, i_k^s).$$

We arrange them in increasing order and rename them as

$$J_k^1 = (j_1^1, \dots, j_k^1), \dots, J_k^s = (j_1^s, \dots, j_k^s).$$

More precisely, we have

$$j_1^1 < \cdots < j_k^1 < j_1^2 < \cdots < j_k^2 < \cdots < j_1^s < \cdots < j_k^s$$

and the set of indices are such that

$$I_k^1 \cup \cdots \cup I_k^s = J_k^1 \cup \cdots \cup J_k^s.$$

Now note that the coefficient of

$$e^{J_k^1} \wedge \cdots \wedge e^{J_k^s} = e^{j_1^1} \wedge \cdots \wedge e^{j_k^1} \wedge \cdots \wedge e^{j_1^s} \wedge \cdots \wedge e^{j_k^s}$$

in ξ^s is equal to

$$(s!) \sum_{\sigma} \text{sgn}(\sigma) \xi_{\sigma(J_k^1)} \cdots \xi_{\sigma(J_k^s)}$$

where the sum runs over all allowed σ . Since we assume that (3.19) is true, we can infer from (3.15) that we can define the coefficient of

$$e^{J_k^1} \wedge \cdots \wedge e^{J_k^s}$$

in $c_s \in \Lambda^{ks}$ as

$$\frac{1}{s!} c_{J_k^1 \cdots J_k^s}.$$

The claim then follows.

Step 2.2. Before concluding the proof, we observe that

$$f_s \left(\sum_{i=1}^{s-1} t_i \alpha_i \right) = 0$$

for any $t_i \in \mathbb{R}$ and where α_i is any of the vectors of the standard basis of Λ^k . This is a direct consequence of the fact that f_s is homogeneous of degree s and that

$$\xi = \sum_{i=1}^{s-1} t_i \alpha_i$$

has only $(s-1)$ coefficients that are non-zero.

Step 2.3. We finally establish (3.19) namely

$$c_{\sigma(I_k^1 \cdots I_k^s)} = \text{sgn}(\sigma) c_{I_k^1 \cdots I_k^s}$$

where σ is a permutation that respects the ordering scheme, more precisely if

$$I_k^1 = (i_1^1, \dots, i_k^1), \dots, I_k^s = (i_1^s, \dots, i_k^s)$$

then, for every $1 \leq m \leq s$,

$$\sigma(i_1^m) < \cdots < \sigma(i_k^m) \quad \text{and} \quad \sigma(i_1^1) < \cdots < \sigma(i_1^s).$$

Note that it is enough to prove the result for the case where σ is a k -flip (see A.3 for definitions), since σ respects the ordering, any such permutation can be written as a product of k -flips (not uniquely, but parity is the same for any such decomposition). We want to show

$$c_{I_k^1 \cdots I_k^s} = -c_{\sigma(I_k^1 \cdots I_k^s)}, \tag{3.20}$$

when σ is a k -flip.

Since σ is a k -flip, we have that σ flips two indices $i_{r_1}^{q_1}$ and $i_{r_2}^{q_2}$, with $q_1 \neq q_2$. Note that, from (3.15), we have

$$c_{I_k^1 \dots I_k^s} = f_s \left(\sum_{m=1}^s e^{I_k^m} \right) = f_s \left(\sum_{m=1}^s e^{i_1^m} \wedge \dots \wedge e^{i_k^m} \right) \quad (3.21)$$

and

$$c_{\sigma(I_k^1 \dots I_k^s)} = f_s \left(\sum_{m=1}^s e^{\sigma(I_k^m)} \right) = f_s \left(\sum_{m=1}^s e^{\sigma(i_1^m)} \wedge \dots \wedge e^{\sigma(i_k^m)} \right). \quad (3.22)$$

We next apply Corollary 3.18 with f_s in place of f (recall that f_s is ext. one affine),

$$a = e^{i_{r_1}^{q_1}}, \quad b = e^{i_{r_2}^{q_2}}, \quad \xi = \sum_{\substack{m=1 \\ m \neq q_1, q_2}}^s e^{i_1^m} \wedge \dots \wedge e^{i_k^m},$$

$$\alpha = \pm e^{i_1^{q_1}} \wedge \dots \wedge \widehat{e^{i_{r_1}^{q_1}}} \wedge \dots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta = \pm e^{i_1^{q_2}} \wedge \dots \wedge \widehat{e^{i_{r_2}^{q_2}}} \wedge \dots \wedge e^{i_k^{q_2}}$$

and the signs are chosen in order to have

$$\alpha \wedge a = e^{I_k^{q_1}} = e^{i_1^{q_1}} \wedge \dots \wedge e^{i_k^{q_1}} \quad \text{and} \quad \beta \wedge b = e^{I_k^{q_2}} = e^{i_1^{q_2}} \wedge \dots \wedge e^{i_k^{q_2}}.$$

Note that our choice of a, b, α, β, ξ implies that,

$$f_s(\xi + \alpha \wedge a + \beta \wedge b) = f_s \left(\sum_{m=1}^s e^{i_1^m} \wedge \dots \wedge e^{i_k^m} \right)$$

and

$$f_s(\xi + \beta \wedge a + \alpha \wedge b) = f_s \left(\sum_{m=1}^s e^{\sigma(i_1^m)} \wedge \dots \wedge e^{\sigma(i_k^m)} \right).$$

We therefore obtain

$$\begin{aligned} & [f_s(\xi + \alpha \wedge a + \beta \wedge b) - f_s(\xi)] + [f_s(\xi + \beta \wedge a + \alpha \wedge b) - f_s(\xi)] \\ &= [f_s(\xi + \alpha \wedge a) - f_s(\xi)] + [f_s(\xi + \beta \wedge b) - f_s(\xi)] \\ &+ [f_s(\xi + \beta \wedge a) - f_s(\xi)] + [f_s(\xi + \alpha \wedge b) - f_s(\xi)] \end{aligned}$$

But except for

$$f_s(\xi + \alpha \wedge a + \beta \wedge b) \quad \text{and} \quad f_s(\xi + \beta \wedge a + \alpha \wedge b)$$

all the other terms are 0 by Step 2.2. We therefore find that

$$f_s(\xi + \alpha \wedge a + \beta \wedge b) = -f_s(\xi + \beta \wedge a + \alpha \wedge b)$$

Together with (3.21) and (3.22), this proves (3.20). This concludes the proof of Step 2.3 and thus of the theorem. ■

Of course, we also have the following corresponding theorem.

Theorem 3.22 *Let $0 \leq k \leq n - 1$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$. The following statements are then equivalent.*

- (i) *f is int. polyaffine.*
- (ii) *f is int. quas affine.*
- (iii) *f is int. one affine.*
- (iv) *There exist $d_r \in \Lambda^{(n-k)r}$, $0 \leq r \leq \left\lfloor \frac{n}{n-k} \right\rfloor$, such that, for every $\eta \in \Lambda^k$,*

$$f(\eta) = \sum_{r=0}^{\left\lfloor \frac{n}{n-k} \right\rfloor} \langle d_r; (*\eta)^r \rangle.$$

Remark 3.23 *As before, once again these functions are precisely the ‘Null Lagrangians’, in the terminology of Ball [4], in this context.*

Proof By virtue of theorem 3.5, f is int. one affine if and only if f_* is ext. one affine and the theorem follows using theorem 3.20. ■

3.4 Examples

3.4.1 The quadratic case

The special case when $f : \Lambda^k \rightarrow \mathbb{R}$ is a quadratic form on Λ^k deserves a special attention.

Some preliminary results

Before stating the main theorem on quadratic forms, we need a lemma. The proof of this lemma is exactly analogous to Lemma 5.27 in [25] and is omitted.

Lemma 3.24 *Let $1 \leq k \leq n$, $M : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^n)$ be a symmetric linear operator and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be such that, for every $\xi \in \Lambda^k(\mathbb{R}^n)$,*

$$f(\xi) = \langle M\xi; \xi \rangle.$$

The following statements then hold true.

- (i) *f is ext. polyconvex if and only if there exists $\beta \in \Lambda^{2k}(\mathbb{R}^n)$ so that, for every $\xi \in \Lambda^k(\mathbb{R}^n)$,*

$$f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle.$$

- (ii) *f is ext. quasiconvex if and only if*

$$\int_{\Omega} f(d\omega) \geq 0$$

for every bounded open set Ω and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^{k-1})$.

(iii) f is ext. one convex if and only if

$$f(a \wedge b) \geq 0$$

for every $a \in \Lambda^{k-1}(\mathbb{R}^n)$ and $b \in \Lambda^1(\mathbb{R}^n)$.

Remark 3.25 Clearly, f is convex if and only if $f(\xi) \geq 0$ for every $\xi \in \Lambda^k$.

Some examples in the quadratic case

We recall that a k form α is said to be 1-divisible if there exist $a \in \Lambda^{k-1}$ and $b \in \Lambda^1$ such that

$$\alpha = a \wedge b.$$

Proposition 3.26 Let $2 \leq k \leq n-2$. Let $\alpha \in \Lambda^k(\mathbb{R}^n)$ be not 1-divisible, then there exists $c > 0$ such that

$$f(\xi) = |\xi|^2 - c(\langle \alpha; \xi \rangle)^2$$

is ext. quasiconvex but not convex. If, in addition $\alpha \wedge \alpha = 0$, then the above f , for an appropriate c , is ext. quasiconvex but not ext. polyconvex.

Remark 3.27 (i) It is easy to see that α is not 1-divisible if and only if

$$\text{rank}_1[*\alpha] = n.$$

This results from Remark 2.44 (iv) (with the help of Proposition 2.33 (iii)) in [21]. Such an α always exists if either of the following holds (see Propositions 2.37 (ii) and 2.43 in [21])

- $k = 2$ or $k = n-2$ and $n \geq 4$ is even,
- $3 \leq k \leq n-3$ (this, in particular, implies $n \geq 6$).

For example

$$\alpha = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6 \in \Lambda^3(\mathbb{R}^6)$$

is not 1-divisible.

(ii) Note that when $k = 2$ every form α such that $\alpha \wedge \alpha = 0$ is necessarily 1-divisible. While, as soon as k is even and $4 \leq k \leq n-2$, there exists α not 1-divisible and such that $\alpha \wedge \alpha = 0$; for example

$$\alpha = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6 \in \Lambda^4(\mathbb{R}^6).$$

Proof Since the function is quadratic, the notions of ext. one convexity and ext. quasiconvexity are equivalent (see Theorem 3.30 below). We therefore only need to discuss the ext. one convexity. We divide the proof into two steps.

Step 1. We first show that if

$$\frac{1}{c} = \sup_{a \in \Lambda^{k-1}, b \in \Lambda^1} \left\{ (\langle \alpha; a \wedge b \rangle)^2 : |a \wedge b| = 1 \right\}$$

then

$$\frac{1}{c} < |\alpha|^2.$$

We prove this statement as follows. Let $a_s \in \Lambda^{k-1}$, $b_s \in \Lambda^1$ be a maximizing sequence. Up to a subsequence that we do not relabel we find that there exists $\lambda \in \Lambda^k$ so that

$$a_s \wedge b_s \rightarrow \lambda \quad \text{with} \quad |\lambda| = 1.$$

Similarly, up to a subsequence that we do not relabel, we have that there exists $\bar{b} \in \Lambda^1$ so that

$$\frac{b_s}{|b_s|} \rightarrow \bar{b}.$$

Since

$$a_s \wedge b_s \wedge \frac{b_s}{|b_s|} = 0$$

we deduce that

$$\lambda \wedge \bar{b} = 0.$$

Appealing to Cartan lemma (see Theorem 2.42 in [21]), we find that there exists $\bar{a} \in \Lambda^{k-1}$ such that

$$\lambda = \bar{a} \wedge \bar{b} \quad \text{with} \quad |\bar{a} \wedge \bar{b}| = 1.$$

We therefore have found that

$$\frac{1}{c} = (\langle \alpha; \bar{a} \wedge \bar{b} \rangle)^2.$$

Note that $\frac{1}{c} < |\alpha|^2$ otherwise $\bar{a} \wedge \bar{b}$ would be parallel to α and thus α would be 1-divisible which contradicts the hypothesis.

Step 2. So let

$$f(\xi) = |\xi|^2 - c(\langle \alpha; \xi \rangle)^2.$$

(i) Observe that f is not convex since $c|\alpha|^2 > 1$ (by Step 1) and

$$f(t\alpha) = t^2|\alpha|^2(1 - c|\alpha|^2).$$

(ii) However f is ext. one convex (and thus, invoking part (i) of Theorem 3.30, f is ext. quasiconvex). Indeed let

$$g(t) = f(\xi + ta \wedge b) = |\xi + ta \wedge b|^2 - c(\langle \alpha; \xi + ta \wedge b \rangle)^2.$$

Note that

$$g''(t) = 2 \left[|a \wedge b|^2 - c(\langle \alpha; a \wedge b \rangle)^2 \right]$$

which is non-negative by Step 1. Thus g is convex.

(iii) Let $\alpha \wedge \alpha = 0$ and assume, for the sake of contradiction, that f is ext. polyconvex. Then there must exist (cf. Lemma 3.24) $\beta \in \Lambda^{2k}$ so that, for every $\xi \in \Lambda^k$,

$$f(\xi) \geq \langle \beta; \xi \wedge \xi \rangle.$$

This is clearly impossible, in view of the fact that $c|\alpha|^2 > 1$, since choosing $\xi = \alpha$, we get

$$f(\alpha) = |\alpha|^2 (1 - c|\alpha|^2) < 0 = \langle \beta; \alpha \wedge \alpha \rangle.$$

The proof is therefore complete. ■

We conclude with another example.

Proposition 3.28 *Let $1 \leq k \leq n$, $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a symmetric linear operator and $T^* : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^n)$ be the pullback of T . Let $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be defined, for every $\xi \in \Lambda^k$, by*

$$f(\xi) = \langle T^*(\xi); \xi \rangle.$$

Then f is ext. one convex if and only if f is convex.

Proof Since convexity implies ext. one convexity, we only have to prove the reverse implication.

Step 1. Since T is symmetric, we can find eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ (not necessarily distinct) with a corresponding set of orthonormal eigenvectors $\{\varepsilon^1, \dots, \varepsilon^n\}$. Let $\{e^1, \dots, e^n\}$ be the standard basis of \mathbb{R}^n . Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and Q be the orthogonal matrix so that

$$Q^*(\varepsilon^i) = e^i, \quad \text{for } i = 1, \dots, n.$$

In terms of matrices what we have written just means that

$$T = Q\Lambda Q^t.$$

Observe that, for every $i = 1, \dots, n$,

$$\begin{aligned} T^*(\varepsilon^i) &= (Q\Lambda Q^t)^*(\varepsilon^i) = (Q^t)^*(\Lambda^*(Q^*(\varepsilon^i))) = (Q^t)^*(\Lambda^*(e^i)) \\ &= (Q^t)^*(\lambda_i e^i) = \lambda_i (Q^t)^*(e^i) = \lambda_i \varepsilon^i. \end{aligned}$$

This implies, for every $1 \leq k \leq n$ and $I \in \mathcal{T}^k$,

$$T^*(\varepsilon^I) = T^*(\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}) = T^*(\varepsilon^{i_1}) \wedge \dots \wedge T^*(\varepsilon^{i_k}) = \left(\prod_{j=1}^k \lambda_{i_j} \right) \varepsilon^I.$$

Step 2. Since f is ext one convex and in view of Lemma 3.24 (iii), we have

$$f(\varepsilon^I) = \langle (T^*(\varepsilon^I)); \varepsilon^I \rangle \geq 0$$

and thus

$$\prod_{j=1}^k \lambda_{i_j} = \prod_{i \in I} \lambda_i \geq 0. \tag{3.23}$$

Writing ξ in the basis $\{\varepsilon^1, \dots, \varepsilon^n\}$, we get

$$\begin{aligned} f(\xi) &= \langle T^*(\xi); \xi \rangle = \left\langle T^* \left(\sum_{I \in \mathcal{T}^k} \xi_I \varepsilon^I \right); \sum_{I \in \mathcal{T}^k} \xi_I \varepsilon^I \right\rangle \\ &= \left\langle \sum_{I \in \mathcal{T}^k} \xi_I T^*(\varepsilon^I); \sum_{I \in \mathcal{T}^k} \xi_I \varepsilon^I \right\rangle = \left\langle \sum_{I \in \mathcal{T}^k} \xi_I \left(\prod_{i \in I} \lambda_i \right) \varepsilon^I; \sum_{I \in \mathcal{T}^k} \xi_I \varepsilon^I \right\rangle \\ &= \sum_{I \in \mathcal{T}^k} \left(\prod_{i \in I} \lambda_i \right) (\xi_I)^2 \end{aligned}$$

which according to (3.23) is non negative. This shows that f is convex as wished. ■

A counterexample for $k = 2$ in the quadratic case

Theorem 3.29 *Let $n \geq 6$. Then there exists a quadratic form $f : \Lambda^2(\mathbb{R}^n) \rightarrow \mathbb{R}$ ext. one convex but not ext. polyconvex.*

Proof We first prove that it is enough to establish the theorem for $n = 6$. Assume that we already constructed an ext. one convex function $g : \Lambda^2(\mathbb{R}^6) \rightarrow \mathbb{R}$ which is not ext. polyconvex. In particular (cf. Proposition 3.16 (iii)) there exist $(t_l, \eta_l) \in \mathbb{R}_+ \times \Lambda^2(\mathbb{R}^6)$ with $\sum t_l = 1$ so that

$$\sum t_l g(\eta_l) < g\left(\sum t_l \eta_l\right) \quad \text{and} \quad \sum t_l \eta_l^s = \left(\sum t_l \eta_l\right)^s, \quad s = 2, 3.$$

Define then $\sigma : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^6)$ to be

$$\sigma(\xi) = \sigma \left(\sum_{1 \leq i < j \leq n} \xi_{ij} e^i \wedge e^j \right) = \sum_{1 \leq i < j \leq 6} \xi_{ij} e^i \wedge e^j, \quad \text{for } \xi \in \Lambda^2(\mathbb{R}^n).$$

Finally let

$$f(\xi) = g(\sigma(\xi)).$$

This function is clearly ext. one convex, since g is so. It is also not ext. polyconvex, since choosing $\xi_l \in \Lambda^2(\mathbb{R}^n)$ so that $\xi_l = \eta_l$ (i.e. all the components of ξ_l appearing in $e^i \wedge e^j$ are 0 whenever one of the i, j is larger or equal 7), we get that $\sigma(\xi_l) = \xi_l = \eta_l$ (note that $\xi_l^s = 0$ whenever $s \geq 4$),

$$\sum t_l g(\sigma(\xi_l)) < g\left(\sigma\left(\sum t_l \xi_l\right)\right) \quad \text{and} \quad \sum t_l \xi_l^s = \left(\sum t_l \xi_l\right)^s, \quad s = 2, \dots, \left[\frac{n}{2}\right].$$

So from now on we assume that $n = 6$. Our counterexample is inspired by Serre [59] and Terpstra [69] (see Theorem 5.25 (iii) in [25]). It is more convenient to write here $\xi \in \Lambda^2(\mathbb{R}^6)$ as

$$\xi = \sum_{1 \leq i < j \leq 6} \xi_j^i e^i \wedge e^j.$$

So let

$$g(\xi) = (\xi_2^1)^2 + (\xi_3^1)^2 + (\xi_3^2)^2 + (\xi_5^4)^2 + (\xi_6^4)^2 + (\xi_6^5)^2 + h(\xi)$$

where

$$h(\xi) = (\xi_4^1 - \xi_5^3 - \xi_6^2)^2 + (\xi_5^1 - \xi_4^3 + \xi_6^1)^2 + (\xi_4^2 - \xi_4^3 - \xi_6^1)^2 + (\xi_5^2)^2 + (\xi_6^3)^2.$$

Note that $g \geq 0$. We claim that there exists $\gamma > 0$ so that

$$f(\xi) = g(\xi) - \gamma |\xi|^2$$

is ext. one convex (cf. Step 1) but not ext polyconvex (cf. Step 2).

Step 1. Define

$$\gamma = \inf \{g(a \wedge b) : a, b \in \Lambda^1(\mathbb{R}^6), |a \wedge b| = 1\}.$$

We claim that $\gamma > 0$. This will imply the ext one convexity of

$$f(\xi) = g(\xi) - \gamma |\xi|^2.$$

We proceed by contradiction and assume that $\gamma = 0$. This implies that we can find $a, b \in \Lambda^1(\mathbb{R}^6)$ with $|a \wedge b| = 1$ such that

$$\begin{cases} a^1 b_2 - a^2 b_1 = 0 \\ a^1 b_3 - a^3 b_1 = 0 \\ a^2 b_3 - a^3 b_2 = 0 \end{cases} \quad \begin{cases} a^4 b_5 - a^5 b_4 = 0 \\ a^4 b_6 - a^6 b_4 = 0 \\ a^5 b_6 - a^6 b_5 = 0 \end{cases} \quad \begin{cases} a^2 b_5 - a^5 b_2 = 0 \\ a^3 b_6 - a^6 b_3 = 0 \end{cases}$$

$$\begin{cases} (a^1 b_4 - a^4 b_1) - (a^3 b_5 - a^5 b_3) - (a^2 b_6 - a^6 b_2) = 0 \\ (a^1 b_5 - a^5 b_1) - (a^3 b_4 - a^4 b_3) + (a^1 b_6 - a^6 b_1) = 0 \\ (a^2 b_4 - a^4 b_2) - (a^3 b_4 - a^4 b_3) - (a^1 b_6 - a^6 b_1) = 0. \end{cases}$$

Let us introduce some notation, we write

$$\underline{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \bar{a} = \begin{pmatrix} a^4 \\ a^5 \\ a^6 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix}.$$

Note that the first and second sets of equations lead to

$$\underline{a} \parallel \underline{b} \quad \text{and} \quad \bar{a} \parallel \bar{b}.$$

We consider two cases starting with the generic case.

Case 1: there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\underline{a} = \lambda \underline{b} \quad \text{and} \quad \bar{a} = \mu \bar{b}.$$

(The same reasoning applies to the case $\underline{b} = \lambda \underline{a}$ and $\bar{b} = \mu \bar{a}$). Note that $\lambda \neq \mu$, otherwise we would have $a = \lambda b$ and thus $a \wedge b = 0$ contradicting the fact that $|a \wedge b| = 1$. Inserting this in

the third and fourth sets of equations we get

$$\begin{cases} (\lambda - \mu) b_2 b_5 = 0 \\ (\lambda - \mu) b_3 b_6 = 0 \end{cases} \quad \begin{cases} (\lambda - \mu) [b_1 b_4 - b_3 b_5 - b_2 b_6] = 0 \\ (\lambda - \mu) [b_1 b_5 - b_3 b_4 + b_1 b_6] = 0 \\ (\lambda - \mu) [b_2 b_4 - b_3 b_4 - b_1 b_6] = 0 \end{cases}$$

and thus, since $\lambda \neq \mu$,

$$\begin{cases} b_2 b_5 = 0 \\ b_3 b_6 = 0 \end{cases} \quad \begin{cases} b_1 b_4 - b_3 b_5 - b_2 b_6 = 0 \\ b_1 b_5 - b_3 b_4 + b_1 b_6 = 0 \\ b_2 b_4 - b_3 b_4 - b_1 b_6 = 0 \end{cases}$$

We have to consider separately the cases $b_2 = b_3 = 0$, $b_5 = b_6 = 0$, $b_2 = b_6 = 0$ and $b_3 = b_5 = 0$.

Case 1.1: $b_2 = b_3 = 0$. We thus have

$$\begin{cases} b_1 b_4 = 0 \\ b_1 b_5 + b_1 b_6 = 0 \\ b_1 b_6 = 0. \end{cases}$$

So either $b_1 = 0$ and thus $\underline{b} = 0$ and hence $\underline{a} = 0$ and again this implies that $a = \mu b$ which contradicts the fact that $|a \wedge b| = 1$. Or $b_4 = b_5 = b_6 = 0$ and thus $\bar{b} = \bar{a} = 0$ which as before contradicts the fact that $|a \wedge b| = 1$.

Case 1.2: $b_5 = b_6 = 0$. This is handled as before. More precisely

$$\begin{cases} b_1 b_4 = 0 \\ b_3 b_4 = 0 \\ b_2 b_4 - b_3 b_4 = 0 \end{cases}$$

Either $b_4 = 0$ and thus $\bar{b} = \bar{a} = 0$ which as before contradicts the fact that $|a \wedge b| = 1$. Or $b_1 = b_2 = b_3 = 0$ and the same contradiction holds.

Case 1.3: $b_2 = b_6 = 0$. We thus have

$$\begin{cases} b_1 b_4 - b_3 b_5 = 0 \\ b_1 b_5 - b_3 b_4 = 0 \\ b_3 b_4 = 0. \end{cases}$$

So either $b_3 = 0$ and we are back in Case 1.1 or $b_4 = 0$ and thus $b_3 b_5 = b_1 b_5 = 0$ and this time we are in Case 1.2.

Case 1.4: $b_3 = b_5 = 0$. We therefore get

$$\begin{cases} b_1 b_4 - b_2 b_6 = 0 \\ b_1 b_6 = 0 \\ b_2 b_4 - b_1 b_6 = 0. \end{cases}$$

Thus either $b_6 = 0$ and we are back in Case 1.2, or $b_1 = 0$ and hence $b_2 b_6 = b_2 b_4 = 0$ which, as before, is impossible.

Case 2: $\underline{b} = 0$ and $\bar{a} = 0$ (or $\underline{a} = 0$ and $\bar{b} = 0$ which is handled similarly). This means that $a^4 = a^5 = a^6 = 0$ and $b_1 = b_2 = b_3 = 0$. We therefore have

$$\begin{cases} a^2 b_5 = 0 \\ a^3 b_6 = 0 \end{cases} \quad \begin{cases} a^1 b_4 - a^3 b_5 - a^2 b_6 = 0 \\ a^1 b_5 - a^3 b_4 + a^1 b_6 = 0 \\ a^2 b_4 - a^3 b_4 - a^1 b_6 = 0. \end{cases}$$

Four cases can happen $a^2 = a^3 = 0$, $a^2 = b_6 = 0$, $a^3 = b_5 = 0$ and $b_5 = b_6 = 0$.

Case 2.1: $a^2 = a^3 = 0$. We thus have

$$\begin{cases} a^1 b_4 = 0 \\ a^1 b_5 + a^1 b_6 = 0 \\ a^1 b_6 = 0. \end{cases}$$

So either $a^1 = 0$ and thus $a = 0$ which is impossible. Or $b_4 = b_5 = b_6 = 0$ and thus $b = 0$ which again cannot happen.

Case 2.2: $a^2 = b_6 = 0$. We thus have

$$\begin{cases} a^1 b_4 - a^3 b_5 = 0 \\ a^1 b_5 - a^3 b_4 = 0 \\ a^3 b_4 = 0 \end{cases}$$

which again cannot happen.

Case 2.3: $a^3 = b_5 = 0$. We thus have

$$\begin{cases} a^1 b_4 - a^2 b_6 = 0 \\ a^1 b_6 = 0 \\ a^2 b_4 - a^1 b_6 = 0. \end{cases}$$

The same reasoning applies also.

Case 2.4: $b_5 = b_6 = 0$. We thus have

$$\begin{cases} a^1 b_4 = 0 \\ a^3 b_4 = 0 \\ a^2 b_4 - a^3 b_4 = 0. \end{cases}$$

As before this is impossible.

Step 2. We now show that f is not ext. polyconvex. In view of Lemma 3.24 (ii) it is sufficient to show that for every $\alpha \in \Lambda^4(\mathbb{R}^6)$, there exists $\xi \in \Lambda^2(\mathbb{R}^6)$ such that

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle < 0.$$

We prove that the above inequality holds for forms ξ of the following form

$$\xi = \sum_{i=1}^3 \sum_{j=4}^6 \xi_j^i e^i \wedge e^j.$$

where

$$\begin{aligned} \xi_4^1 &= b + d, & \xi_5^1 &= c - a, & \xi_6^1 &= a \\ \xi_4^2 &= c + a, & \xi_5^2 &= 0, & \xi_6^2 &= b \\ \xi_4^3 &= c, & \xi_5^3 &= d, & \xi_6^3 &= 0 \end{aligned}$$

all the other ξ_j^i being 0. In other words

$$\begin{aligned} \xi &= (b + d) e^1 \wedge e^4 + (c - a) e^1 \wedge e^5 + (a) e^1 \wedge e^6 \\ &\quad + (c + a) e^2 \wedge e^4 + (b) e^2 \wedge e^6 + (c) e^3 \wedge e^4 + (d) e^3 \wedge e^5. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{2} \xi \wedge \xi &= (c^2 - a^2) e^1 \wedge e^2 \wedge e^4 \wedge e^5 + (ac + a^2 - b^2 - bd) e^1 \wedge e^2 \wedge e^4 \wedge e^6 \\ &\quad + (ab - bc) e^1 \wedge e^2 \wedge e^5 \wedge e^6 + (c^2 - ac - bd - d^2) e^1 \wedge e^3 \wedge e^4 \wedge e^5 \\ &\quad + (ac) e^1 \wedge e^3 \wedge e^4 \wedge e^6 + (ad) e^1 \wedge e^3 \wedge e^5 \wedge e^6 \\ &\quad + (-cd - ad) e^2 \wedge e^3 \wedge e^4 \wedge e^5 + (bc) e^2 \wedge e^3 \wedge e^4 \wedge e^6 \\ &\quad + (bd) e^2 \wedge e^3 \wedge e^5 \wedge e^6 \end{aligned}$$

For such forms we have $g(\xi) = 0$ and therefore

$$f(\xi) = -\gamma |\xi|^2 = -\gamma \left[(b + d)^2 + (c - a)^2 + a^2 + (c + a)^2 + b^2 + c^2 + d^2 \right]$$

moreover

$$\begin{aligned} \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle &= \alpha_{1245} (c^2 - a^2) + \alpha_{1246} (ac + a^2 - b^2 - bd) \\ &\quad + \alpha_{1256} (ab - bc) + \alpha_{1345} (c^2 - ac - bd - d^2) + \alpha_{1346} (ac) \\ &\quad + \alpha_{1356} (ad) + \alpha_{2345} (-cd - ad) + \alpha_{2346} (bc) + \alpha_{2356} (bd). \end{aligned}$$

We consider three cases.

Case 1. If $\alpha_{1246} > 0$, then take $a = c = d = 0$ and $b \neq 0$, to get

$$\begin{aligned} f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle &= -\gamma |\xi|^2 + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle \\ &= -\gamma (2b^2) - \alpha_{1246} b^2 < 0. \end{aligned}$$

Case 2. If $\alpha_{1345} > 0$, then take $a = b = c = 0$ and $d \neq 0$, to get

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle = -\gamma (2d^2) - \alpha_{1345} d^2 < 0.$$

We therefore can assume that $\alpha_{1246} \leq 0$ and $\alpha_{1345} \leq 0$.

Case 3. If $\alpha_{1245} + \alpha_{1345} < 0$ ($\alpha_{1246} \leq 0$, $\alpha_{1345} \leq 0$), then take $a = b = d = 0$ and $c \neq 0$ to get

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle = -\gamma(3c^2) + (\alpha_{1245} + \alpha_{1345})c^2 < 0.$$

We therefore assume $\alpha_{1246} \leq 0$, $\alpha_{1345} \leq 0$ and $\alpha_{1245} + \alpha_{1345} \geq 0$. From these three inequalities we deduce that $\alpha_{1246} - \alpha_{1245} \leq 0$, and then taking $b = c = d = 0$ and $a \neq 0$, we get

$$f(\xi) + \frac{1}{2} \langle \alpha; \xi \wedge \xi \rangle = -\gamma(3a^2) + (\alpha_{1246} - \alpha_{1245})a^2 < 0.$$

And this concludes the proof of the theorem. ■

The main result for quadratic functions

We now turn to the main theorem.

Theorem 3.30 (Summary of the quadratic case) *Let $1 \leq k \leq n$, $M : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^n)$ be a symmetric linear operator and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be such that, for every $\xi \in \Lambda^k(\mathbb{R}^n)$,*

$$f(\xi) = \langle M\xi; \xi \rangle.$$

(i) *The following equivalence holds in all cases*

$$f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(ii) *Let $k = 2$. If $n = 2$ or $n = 3$, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

If $n = 4$, then

$$f \text{ convex} \xRightarrow{\nRightarrow} f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex}$$

while if $n \geq 6$, then

$$f \text{ ext. polyconvex} \xRightarrow{\nRightarrow} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

(iii) *If k is odd or if $2k > n$, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iv) *If k is even and $2k \leq n$, then*

$$f \text{ convex} \xRightarrow{\nRightarrow} f \text{ ext. polyconvex.}$$

(v) If either $3 \leq k \leq n - 3$ or $k = n - 2 \geq 4$ is even, then

$$f \text{ ext. polyconvex} \xRightarrow{\quad} f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex}.$$

Remark 3.31 (i) We recall that when $k = 1$ all notions of convexity are equivalent.

(ii) When $k = 2$ and $n = 5$, the equivalence between polyconvexity and quasiconvexity remains open.

Proof (i) The result follows from Theorem 3.54 and classical results (see Theorem 5.25 in [25]). It can, of course, be proved directly using Fourier transform in a completely analogous manner.

(ii) If $n = 2$ or $n = 3$, the result follows from Theorem 3.12. If $n \geq 6$, see Theorem 3.29. So we now assume that $n = 4$ (for the counter implication see (iv) below). We only have to prove that

$$f \text{ ext. one convex} \Rightarrow f \text{ ext. polyconvex}.$$

We know (by ext. one convexity) that, for every $a, b \in \Lambda^1(\mathbb{R}^4)$

$$f(a \wedge b) \geq 0$$

and we wish to show (cf. Lemma 3.24) that we can find $\alpha \in \Lambda^4(\mathbb{R}^4)$ so that

$$f(\xi) \geq \langle \alpha; \xi \wedge \xi \rangle.$$

Step 1. Let us change slightly the notations and write $\xi \in \Lambda^2(\mathbb{R}^4)$ as a vector of \mathbb{R}^6 in the following manner

$$\xi = (\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34})$$

and therefore f can be seen as a quadratic form over \mathbb{R}^6 which is non-negative whenever the quadratic form (note also that g is indefinite)

$$g(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle = 2(\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23})$$

vanishes. Indeed note that

$$g(\xi) = 0 \Leftrightarrow \xi \wedge \xi = 0 \Leftrightarrow \text{rank}[\xi] = 0, 2$$

This last condition is equivalent to the existence of $a, b \in \Lambda^1(\mathbb{R}^4)$ so that

$$\xi = a \wedge b$$

and by ext. one convexity we know that $f(a \wedge b) \geq 0$.

Step 2. We now invoke Theorem 2 in [47] (see also [37] or [71]) to get that there exists $\lambda \in \mathbb{R}$ such that

$$f(\xi) - \lambda g(\xi) \geq 0.$$

But this is exactly what we had to prove.

(iii) This is a general fact (cf. Theorem 3.16).

(iv) The counterexample is just

$$f(\xi) = \langle \alpha; \xi \wedge \xi \rangle$$

for any $\alpha \in \Lambda^{2k}(\mathbb{R}^n)$, $\alpha \neq 0$.

(v) This is just Proposition 3.26 and the remark following it. Indeed we consider the two following cases.

- If k is odd (and since $3 \leq k \leq n-3$, then $n \geq 6$), then we know from (iii) that f is ext. polyconvex if and only if f is convex and we also know that there exists an α which is not 1-divisible. Proposition 3.26 gives therefore the result.

- If k is even and $4 \leq k \leq n-2$ (which implies again $n \geq 6$), then there exists an α which is not 1-divisible such that $\alpha \wedge \alpha = 0$. The result thus follows again by Proposition 3.26. ■

3.4.2 Ext. quasiconvexity does not imply ext. polyconvexity

We here give another counterexample for $k = 2$.

Proposition 3.32 *Let $n \geq 4$. Then there exists an ext. quasiconvex function over $\Lambda^2(\mathbb{R}^n)$ which is not ext. polyconvex.*

Remark 3.33 *This example is mostly interesting when $n = 4$ or 5. Since when $n \geq 6$, we already have such a counterexample (cf. Theorem 3.29).*

Proof As in previous theorems it is easy to see that it is enough to establish the theorem for $n = 4$. Let $1 < p < 2$, $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4$ and $g : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$ be given by

$$g(\xi) = \left(|\xi|^2 - 2|\langle \alpha; \xi \rangle| + |\alpha|^2 \right)^{p/2} = \min \{ |\xi - \alpha|^p, |\xi + \alpha|^p \}.$$

The claim is that $f = Q_{ext}g$ has all the desired properties (the proof is inspired by the one of Šverák [63], see also Theorem 5.54 in [25]). Indeed f is by construction ext. quasiconvex and if we can show (cf. Step 2) that f is not convex (here since the function f grows less than quadratically ext. polyconvexity and convexity are equivalent) we will have established the proposition.

Step 1. First observe that a direct computation gives

$$|\xi|^2 - 2|\langle \alpha; \xi \rangle| + |\alpha|^2 = \min \{ |\xi - \alpha|^2, |\xi + \alpha|^2 \} \geq \frac{1}{2} \left[|\xi|^2 - \frac{1}{2} \langle \alpha \wedge \alpha; \xi \wedge \xi \rangle \right] \geq 0.$$

We therefore get that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} g(\xi) &\geq \frac{1}{2} \left[|\xi|^2 - \frac{1}{2} \langle \alpha \wedge \alpha; \xi \wedge \xi \rangle \right]^{p/2} \\ &\geq c_1 [|\xi_{12} - \xi_{34}|^p + |\xi_{13} + \xi_{24}|^p + |\xi_{14} - \xi_{23}|^p]. \end{aligned}$$

Call h the right hand side, namely

$$h(\xi) = c_1 [|\xi_{12} - \xi_{34}|^p + |\xi_{13} + \xi_{24}|^p + |\xi_{14} - \xi_{23}|^p].$$

Step 2. Note that if f were convex (clearly $f \geq 0$), we should have

$$0 \leq f(0) = f\left(\frac{1}{2}\alpha + \frac{1}{2}(-\alpha)\right) \leq \frac{1}{2}f(\alpha) + \frac{1}{2}f(-\alpha) = 0.$$

We however will show that

$$f(0) > 0$$

and this will establish the proposition. We proceed by contradiction and assume that

$$f(0) = 0.$$

Use the remark following Proposition 3.14 to find a sequence of $\omega_s \in W_{\delta,T}^{1,\infty}(\Omega; \Lambda^1)$ (we can choose an Ω with smooth boundary and by density we can also assume that $\omega_s \in C_{\delta,T}^\infty(\overline{\Omega}; \Lambda^1)$) such that

$$0 \leq \frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) \leq Q_{ext}g(0) + \frac{1}{s} = f(0) + \frac{1}{s} = \frac{1}{s}.$$

From Step 1, we deduce that

$$0 \leq \int_{\Omega} h(d\omega_s) \leq \frac{\text{meas } \Omega}{s} \rightarrow 0.$$

We now invoke Step 3 to get that there exists a constant $c_2 > 0$ such that

$$c_2 \|\nabla \omega_s\|_{L^p}^p \leq \int_{\Omega} h(d\omega_s).$$

Thus $\|d\omega_s\|_{L^p} \rightarrow 0$ and hence, up to the extraction of a subsequence,

$$\frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) \rightarrow g(0) = |\alpha|^p \neq 0.$$

Since at the same time

$$\frac{1}{\text{meas } \Omega} \int_{\Omega} g(d\omega_s) \rightarrow Q_{ext}g(0) = f(0) = 0$$

we have obtained the desired contradiction.

Step 3. It remains to prove that there exists a constant $\lambda > 0$ such that

$$\lambda \|\nabla \omega\|_{L^p}^p \leq \int_{\Omega} h(d\omega) = \left\| [h(d\omega)]^{1/p} \right\|_{L^p}, \quad \text{for every } \omega \in C_{\delta,T}^\infty(\overline{\Omega}; \Lambda^1).$$

To establish the estimate we proceed as follows. Let $\omega \in C_{\delta,T}^\infty(\bar{\Omega}; \Lambda^1)$, $\alpha, \beta, \gamma \in C^\infty(\bar{\Omega})$ be such that

$$\begin{aligned}\alpha &= (d\omega)_{12} - (d\omega)_{34} = -\omega_{x_2}^1 + \omega_{x_1}^2 + \omega_{x_4}^3 - \omega_{x_3}^4 \\ \beta &= (d\omega)_{13} + (d\omega)_{24} = -\omega_{x_3}^1 + \omega_{x_1}^3 - \omega_{x_4}^2 + \omega_{x_2}^4 \\ \gamma &= (d\omega)_{14} - (d\omega)_{23} = -\omega_{x_4}^1 + \omega_{x_1}^4 + \omega_{x_3}^2 - \omega_{x_2}^3 \\ 0 &= \delta\omega = \omega_{x_1}^1 + \omega_{x_2}^2 + \omega_{x_3}^3 + \omega_{x_4}^4.\end{aligned}$$

Note that

$$h(d\omega) = c_1[|\alpha|^p + |\beta|^p + |\gamma|^p].$$

Differentiating appropriately the four equations we find

$$\begin{aligned}\Delta\omega^1 &= -\alpha_{x_2} - \beta_{x_3} - \gamma_{x_4} \\ \Delta\omega^2 &= \alpha_{x_1} - \beta_{x_4} + \gamma_{x_3} \\ \Delta\omega^3 &= \alpha_{x_4} + \beta_{x_1} - \gamma_{x_2} \\ \Delta\omega^4 &= -\alpha_{x_3} + \beta_{x_2} + \gamma_{x_1}.\end{aligned}$$

Letting

$$\phi = \alpha dx^1 \wedge dx^2 + \beta dx^1 \wedge dx^3 + \gamma dx^1 \wedge dx^4 - \gamma dx^2 \wedge dx^3 + \beta dx^2 \wedge dx^4 - \alpha dx^3 \wedge dx^4,$$

we get

$$\begin{cases} \Delta\omega = \delta\phi & \text{in } \Omega \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega \\ \nu \wedge \delta\omega = 0 & \text{on } \partial\Omega. \end{cases}$$

and This implies via elliptic regularity of the Hodge Laplacian that

$$\|\nabla\omega\|_{L^p} \leq \lambda_2 \|\phi\|_{L^p}$$

or, in other words,

$$\|\nabla\omega\|_{L^p} \leq \lambda_2 \|\phi\|_{L^p} \leq \lambda_3 \|(\alpha, \beta, \gamma)\|_{L^p} \leq \lambda_4 \left\| [h(d\omega)]^{1/p} \right\|_{L^p}.$$

This is exactly what had to be proved. ■

3.4.3 Ext one convexity does not imply ext quasiconvexity

We now give an important counterexample for any $k \geq 2$. It is an adaptation of the fundamental result of Šverák [65] (see also Theorem 5.50 in [25]), though with nontrivial algebraic manipulations.

Theorem 3.34 *Let $2 \leq k \leq n - 3$. Then there exists $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ ext. one convex but not ext. quasiconvex.*

Remark 3.35 We know that when $k = 1, n - 1, n$ or $k = n - 2$ is odd, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Therefore only the case $k = n - 2 \geq 2$ even (including $k = 2$ and $n = 4$) remains open.

The main algebraic tool in order to adapt Šverák's example is given in the following lemma. This algebraic part is trivial in the Šverák's proof in the classical case.

Lemma 3.36 Let $k \geq 2$ and $n = k + 3$. There exist

$$\alpha, \beta, \gamma \in \text{span} \{e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}, \quad 3 \leq i_1 < \cdots < i_{k-1} \leq k + 3\} \subset \Lambda^{k-1}(\mathbb{R}^{k+1})$$

such that if

$$L = \text{span} \{e^1 \wedge \alpha, e^2 \wedge \beta, (e^1 + e^2) \wedge \gamma\}$$

i.e.

$$L = \left\{ \begin{array}{l} \xi \in \Lambda^k(\mathbb{R}^{k+3}) : \\ \xi = x e^1 \wedge \alpha + y e^2 \wedge \beta + z (e^1 + e^2) \wedge \gamma \\ \quad = e^1 \wedge (x \alpha + z \gamma) + e^2 \wedge (y \beta + z \gamma) \\ x, y, z \in \mathbb{R} \end{array} \right\}$$

and we write, as a shorthand, any $\xi \in L$ as $\xi = (x, y, z)$, then any 1-divisible $\xi = (x, y, z) \in L$ (meaning that $\xi = a \wedge b$ for a certain $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$), necessarily verifies

$$xy = xz = yz = 0.$$

We now establish Lemma 3.36.

Proof Step 1. We choose, recall that $n = k + 3$,

$$\begin{aligned} \alpha &= \begin{cases} \sum_{i=2}^{l+1} (\widehat{e^{2i}} \wedge \widehat{e^{2i+1}}) & \text{if } k = 2l \\ \sum_{i=2}^{l+2} (\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}) & \text{if } k = 2l + 1 \end{cases} \\ \beta &= \begin{cases} \widehat{e^3} \wedge \widehat{e^{2l+3}} & \text{if } k = 2l \\ (\widehat{e^3} \wedge \widehat{e^5}) + (\widehat{e^4} \wedge \widehat{e^6}) & \text{if } k = 3 \\ \sum_{i=2}^l (\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}) & \text{if } k = 2l + 1 \text{ and } k \geq 5 \end{cases} \\ \gamma &= \begin{cases} \sum_{i=2}^{l+1} (\widehat{e^{2i-1}} \wedge \widehat{e^{2i}}) & \text{if } k = 2l \\ (\widehat{e^{2l+1}} \wedge \widehat{e^{2l+4}}) + (\widehat{e^{2l+2}} \wedge \widehat{e^{2l+3}}) & \text{if } k = 2l + 1 \end{cases} \end{aligned}$$

where we write, by abuse of notations,

$$\widehat{e^i} \wedge \widehat{e^j} = e^3 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge \widehat{e^j} \wedge \cdots \wedge e^{k+3}.$$

Observe that $\{\alpha, \beta, \gamma\}$ are linearly independent.

Step 2. We now prove the statement, namely that if $\xi = (x, y, z) \in L$ is 1-divisible (i.e. $\xi = b \wedge a$ for $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$), then necessarily

$$xy = xz = yz = 0.$$

Assume that $\xi \neq 0$ (otherwise the result is trivial) and thus $a \neq 0$. Note that if $\xi = b \wedge a$, then $a \wedge \xi = 0$. We write

$$a = \sum_{i=1}^{k+3} a_i e^i \neq 0.$$

Step 2.1. Since $a \wedge \xi = 0$ we deduce that the term involving $e^1 \wedge e^2$ must be 0 and thus

$$-a_2 x \alpha + a_1 y \beta + (a_1 - a_2) z \gamma = 0.$$

Since $\{\alpha, \beta, \gamma\}$ are linearly independent, we deduce that

$$a_2 x = a_1 y = (a_1 - a_2) z = 0.$$

From there we infer that $xy = xz = yz = 0$, as soon as either $a_1 \neq 0$ or $a_2 \neq 0$. So in order to establish the lemma it is enough to consider a of the form

$$a = \sum_{i=3}^{k+3} a_i e^i \neq 0.$$

We therefore have

$$\sum_{i=3}^{k+3} a_i e^i \wedge [e^1 \wedge (x \alpha + z \gamma) + e^2 \wedge (y \beta + z \gamma)] = 0$$

which implies that

$$\begin{cases} a \wedge (x \alpha + z \gamma) = \sum_{i=3}^{k+3} a_i e^i \wedge (x \alpha + z \gamma) = 0 \\ a \wedge (y \beta + z \gamma) = \sum_{i=3}^{k+3} a_i e^i \wedge (y \beta + z \gamma) = 0. \end{cases} \quad (3.24)$$

We continue the discussion considering separately the cases k even, $k = 3$ and $k \geq 5$ odd. They are all treated in the same way and we prove it only in the even case.

Step 2.2: $k = 2l \geq 2$. We have to prove that if

$$a = \sum_{i=3}^{2l+3} a_i e^i \neq 0$$

satisfies (3.24), then necessarily

$$xy = xz = yz = 0.$$

We find (up to a + or – sign but here it is immaterial)

$$a \wedge \alpha = \sum_{i=2}^{l+1} \left(a_{2i+1} \widehat{e^{2i}} \right) + \sum_{i=2}^l \left(a_{2i} \widehat{e^{2i+1}} \right) + a_{2l+2} \widehat{e^{2l+3}}$$

$$a \wedge \beta = a_{2l+3} \widehat{e^3} + a_3 \widehat{e^{2l+3}}$$

$$a \wedge \gamma = a_4 \widehat{e^3} + \sum_{i=2}^{l+1} \left(a_{2i-1} \widehat{e^{2i}} \right) + \sum_{i=2}^l \left(a_{2i+2} \widehat{e^{2i+1}} \right).$$

Therefore

$$a \wedge (x \alpha + z \gamma) = z a_4 \widehat{e^3} + \sum_{i=2}^{l+1} (x a_{2i+1} + z a_{2i-1}) \widehat{e^{2i}} + \sum_{i=2}^l (x a_{2i} + z a_{2i+2}) \widehat{e^{2i+1}} + x a_{2l+2} \widehat{e^{2l+3}}$$

$$a \wedge (y \beta + z \gamma) = (y a_{2l+3} + z a_4) \widehat{e^3} + z \left\{ \sum_{i=2}^{l+1} \left(a_{2i-1} \widehat{e^{2i}} \right) + \sum_{i=2}^l \left(a_{2i+2} \widehat{e^{2i+1}} \right) \right\} + y a_3 \widehat{e^{2l+3}}.$$

Case 1 : $x = z = 0$. This is our claim.

Case 2 : $z = 0$ and $x \neq 0$. We can also assume that $y \neq 0$ otherwise we have the claim $y = z = 0$. From the first equation we obtain

$$a_{2i} = 0, \quad i = 2, \dots, l+1$$

$$a_{2i+1} = 0, \quad i = 2, \dots, l+1.$$

So only a_3 might be non-zero. However since $y \neq 0$ we deduce from the second equation that $a_3 = 0$ and thus $a = 0$ which is impossible.

Case 3 : $x = 0$ and $z \neq 0$. We can also assume that $y \neq 0$ otherwise we have the claim $x = y = 0$. From the first equation we obtain

$$a_{2i} = 0, \quad i = 2, \dots, l+1$$

$$a_{2i-1} = 0, \quad i = 2, \dots, l+1.$$

So only a_{2l+3} might be non-zero. However since $y \neq 0$ we deduce, appealing to the second equation, that $a_{2l+3} = 0$ and thus $a = 0$ which is again impossible.

Case 4 : $xz \neq 0$. From the first equation we deduce that

$$a_{2i} = 0, \quad i = 2, \dots, l+1$$

Inserting this in the second equation we get

$$a \wedge (y \beta + z \gamma) = y a_{2l+3} \widehat{e^3} + z \sum_{i=2}^{l+1} \left(a_{2i-1} \widehat{e^{2i}} \right) + y a_3 \widehat{e^{2l+3}}.$$

Since $z \neq 0$, we infer that

$$a_{2i-1} = 0, \quad i = 2, \dots, l+1.$$

So only a_{2l+3} might be non-zero. However returning to the first equation we have

$$x a_{2l+3} = 0.$$

But since $x \neq 0$, we deduce that $a_{2l+3} = 0$ and thus $a = 0$ which is again impossible. This settles the case k even. The odd case is handled in a very similar manner and we leave out the details ■

We may now conclude with the proof of Theorem 3.34, which is, once the above lemma established, almost identical to the proof of Šverák.

Proof Preliminary step. We prove here that it is enough to establish the theorem for $n = k+3$. Assume that we already constructed an ext. one convex function $g : \Lambda^k(\mathbb{R}^{k+3}) \rightarrow \mathbb{R}$ which is not ext. quasiconvex. In particular there exists $\eta \in \Lambda^k(\mathbb{R}^{k+3})$ and $\psi \in W_{per}^{1,\infty}(D_{k+3}; \Lambda^{k-1}(\mathbb{R}^{k+3}))$, where $D_n = (0,1)^n$, so that

$$\int_{D_{k+3}} g(\eta + d\psi(x)) dx < g(\eta).$$

Define then $\sigma : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^{k+3})$ to be, for $\xi \in \Lambda^k(\mathbb{R}^n)$,

$$\begin{aligned} \sigma(\xi) &= \sigma \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} \right) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq k+3} \xi_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}. \end{aligned}$$

Finally let

$$f(\xi) = g(\sigma(\xi)).$$

This function is clearly ext. one convex, since g is so. It is also not ext. quasiconvex, since choosing any $\xi \in \Lambda^k(\mathbb{R}^n)$ so that $\sigma(\xi) = \eta$ and

$$\varphi^{i_1 \dots i_{k-1}}(x_1, \dots, x_n) = \begin{cases} \psi^{i_1 \dots i_{k-1}}(x_1, \dots, x_{k+3}) & \text{if } 1 \leq i_1 < \dots < i_{k-1} \leq k+3 \\ 0 & \text{if not} \end{cases}$$

we get that $\varphi \in W_{per}^{1,\infty}(D_n; \Lambda^{k-1}(\mathbb{R}^n))$ and

$$\int_{D_n} f(\xi + d\varphi(x)) dx < f(\xi).$$

So from now on we assume that $n = k+3$.

Step 1. We start with some notations. Let L be as in Lemma 3.36. An element ξ of L is, when convenient, denoted by $\xi = (x, y, z) \in L$. Recall that if $\xi = (x, y, z) \in L$ is 1-divisible,

meaning that $\xi = b \wedge a$ for a certain $a \in \Lambda^1$ and $b \in \Lambda^{k-1}$, then necessarily

$$xy = xz = yz = 0.$$

We next let $P : \Lambda^k(\mathbb{R}^{k+3}) \rightarrow L$ be the projection map; in particular $P(\xi) = \xi$ if $\xi \in L$.

Step 2. Let $g : L \subset \Lambda^k(\mathbb{R}^{k+3}) \rightarrow \mathbb{R}$ be defined by

$$g(\xi) = -xyz.$$

Observe that g is ext. one affine when restricted to L . Indeed if $\xi = (x, y, z) \in L$ and $\eta = (a, b, c) \in L$ is 1-divisible (which implies that $ab = ac = bc = 0$), then

$$\begin{aligned} g(\xi + t\eta) &= -(x + ta)(y + tb)(z + tc) \\ &= -xyz - t[xyz + xzb + yza]. \end{aligned}$$

We therefore have that, for every $\xi, \eta \in L$ with η 1-divisible,

$$L_g(\xi, \eta) = \left. \frac{d^2}{dt^2} g(\xi + t\eta) \right|_{t=0} = 0.$$

Step 3. By abuse of notations we identify the exterior forms $\{\alpha, \beta, \gamma\}$ with differential forms (replacing e^i with dx^i). Let ω be defined by

$$\omega = (\sin x_1) \alpha + (\sin x_2) \beta + (\sin(x_1 + x_2)) \gamma$$

so that $\omega \in C_{per}^\infty((0, 2\pi)^{k+3}; \Lambda^{k-1})$ and

$$d\omega = (\cos x_1) dx^1 \wedge \alpha + (\cos x_2) dx^2 \wedge \beta + (\cos(x_1 + x_2)) (dx^1 + dx^2) \wedge \gamma$$

and hence $d\omega \in L$. Note that

$$\int_0^{2\pi} \int_0^{2\pi} g(d\omega) dx_1 dx_2 = - \int_0^{2\pi} \int_0^{2\pi} (\cos x_1)^2 (\cos x_2)^2 dx_1 dx_2 < 0.$$

Step 4. Assume, cf. Step 5, that we have shown that for every $\epsilon > 0$ we can find $\gamma = \gamma(\epsilon) > 0$ such that

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon |\xi|^2 + \epsilon |\xi|^4 + \gamma |\xi - P(\xi)|^2$$

is ext. one convex. Then noting that

$$f_\epsilon(d\omega) = g(d\omega) + \epsilon |d\omega|^2 + \epsilon |d\omega|^4$$

we deduce from Step 3 that for $\epsilon > 0$ small enough

$$\int_{(0, 2\pi)^{k+3}} f_\epsilon(d\omega) dx < 0.$$

This shows that f_ϵ is not ext. quasiconvex. The proposition is therefore proved.

Step 5. It remains to prove that for every $\epsilon > 0$ we can find $\gamma = \gamma(\epsilon) > 0$ such that

$$f_\epsilon(\xi) = g(P(\xi)) + \epsilon |\xi|^2 + \epsilon |\xi|^4 + \gamma |\xi - P(\xi)|^2$$

is ext. one convex. This is equivalent to showing that, for every $\xi, \eta \in \Lambda^k$ with η 1-divisible,

$$\begin{aligned} L_f(\xi, \eta) &= \left. \frac{d^2}{dt^2} f(\xi + t\eta) \right|_{t=0} \\ &= L_g(P(\xi), P(\eta)) + 2\epsilon |\eta|^2 + 4\epsilon |\xi|^2 |\eta|^2 + 8\epsilon (\langle \xi; \eta \rangle)^2 + 2\gamma |\eta - P(\eta)|^2 \\ &\geq 0. \end{aligned}$$

Step 5.1. Observe that since g is a homogeneous of degree 3 polynomial, we can find $c > 0$ so that

$$L_g(P(\xi), P(\eta)) \geq -c |\xi| |\eta|^2.$$

We therefore deduce that

$$L_f(\xi, \eta) \geq (-c + 4\epsilon |\xi|) |\xi| |\eta|^2$$

and thus $L_f(\xi, \eta) \geq 0$ holds for every $\eta \in \Lambda^k$ (independently of the fact that η is 1-divisible) and for every $\xi \in \Lambda^k$ which satisfies

$$|\xi| \geq \frac{c}{4\epsilon}.$$

Step 5.2. It therefore remains to show that $L_f(\xi, \eta) \geq 0$ in the compact set

$$K = \left\{ (\xi, \eta) \in \Lambda^k \times \Lambda^k : |\xi| \leq \frac{c}{4\epsilon}, |\eta| = 1, \eta \text{ 1-divisible} \right\}$$

in view of Step 5.1 and of the fact that $L_f(\xi, \eta)$ is homogeneous of degree 2 in the variable η . Moreover we also find that

$$L_f(\xi, \eta) \geq H(\xi, \eta, \gamma) = L_g(P(\xi), P(\eta)) + 2\epsilon |\eta|^2 + 2\gamma |\eta - P(\eta)|^2$$

and therefore $L_f(\xi, \eta) \geq 0$ will follow if we can show that for every $\epsilon > 0$ we can find $\gamma = \gamma(\epsilon)$ so that $H \geq 0$ on K . Assume, for the sake of contradiction, that this is not the case. We can then find $\gamma_\nu \rightarrow \infty$, $(\xi_\nu, \eta_\nu) \in K$ so that

$$L_g(P(\xi_\nu), P(\eta_\nu)) + 2\epsilon \leq L_g(P(\xi_\nu), P(\eta_\nu)) + 2\epsilon + 2\gamma_\nu |\eta_\nu - P(\eta_\nu)|^2 < 0.$$

Since K is compact, we have up to a subsequence (still labeled (ξ_ν, η_ν)) that

$$(\xi_\nu, \eta_\nu) \rightarrow (\xi, \eta) \in K, \quad L_g(P(\xi), P(\eta)) + 2\epsilon \leq 0 \quad \text{and} \quad P(\eta) = \eta.$$

However we have $\epsilon > 0$ and, cf. Step 2,

$$L_g(P(\xi), P(\eta)) \equiv 0, \quad \forall \xi, \eta \in \Lambda^k \text{ with } P(\eta) = \eta \text{ where } \eta \text{ is 1-divisible.}$$

This leads to the desired contradiction. ■

3.4.4 Summary of implications and counter-implications

The examples, counter examples and results we have obtained so far gives us an almost complete picture of the relationship between the different notions of convexity. We summarize them in the following theorem.

Theorem 3.37 *Let $1 \leq k \leq n$ and $f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$.*

(i) The following implications then hold

$$f \text{ convex} \Rightarrow f \text{ ext. polyconvex} \Rightarrow f \text{ ext. quasiconvex} \Rightarrow f \text{ ext. one convex.}$$

(ii) If $k = 1, n - 1, n$ or $k = n - 2$ is odd, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

Moreover if k is odd or $2k > n$, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex.}$$

(iii) If either $k = 2$ and $n \geq 4$ or $3 \leq k \leq n - 3$ or $k = n - 2 \geq 4$ is even, then

$$f \text{ ext. polyconvex} \xRightarrow{\quad} f \text{ ext. quasiconvex}$$

while if $2 \leq k \leq n - 3$ (and thus $n \geq k + 3 \geq 5$), then

$$f \text{ ext. quasiconvex} \xRightarrow{\quad} f \text{ ext. one convex.}$$

Remark 3.38 *(i) The study of the implications and counter implications for convexity, polyconvexity and quasiconvexity is therefore complete. For the last implication namely*

$$f \text{ ext. quasiconvex} \xRightarrow{\quad} f \text{ ext. one convex}$$

only the case $k = n - 2 \geq 2$ even (including $k = 2$ and $n = 4$) remains open.

(ii) The last statement in (ii) for k even and $n \geq 2k$ is false, as the following simple example shows. Let $f : \Lambda^2(\mathbb{R}^4) \rightarrow \mathbb{R}$ be defined by

$$f(\xi) = \langle e^1 \wedge e^2 \wedge e^3 \wedge e^4; \xi \wedge \xi \rangle.$$

The function f is clearly ext. polyconvex but not convex.

(iii) It is interesting to read the theorem when $k = 2$.

- If $n = 2$ or $n = 3$, then

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex} \Leftrightarrow f \text{ ext. quasiconvex} \Leftrightarrow f \text{ ext. one convex.}$$

- If $n \geq 4$, then

$$f \text{ convex} \xRightarrow{\neq} f \text{ ext. polyconvex} \xRightarrow{\neq} f \text{ ext. quasiconvex}.$$

- If $n \geq 5$, then

$$f \text{ ext. quasiconvex} \xRightarrow{\neq} f \text{ ext. one convex}$$

while the case $n = 4$ remains open.

Proof (i) This conclusion is exactly Theorem 3.10.

(ii) The first statement is just Theorem 3.12. The extra statement (i.e. when k is odd or $2k > n$)

$$f \text{ convex} \Leftrightarrow f \text{ ext. polyconvex}.$$

is proved in Proposition 3.16 (iii).

(iii) The statement that

$$f \text{ ext. polyconvex} \xRightarrow{\neq} f \text{ ext. quasiconvex}$$

when $3 \leq k \leq n-3$ or $k = n-2 \geq 4$ is even follows from Theorem 3.30 (v) and from Proposition 3.32 when $k = 2$ and $n \geq 4$ (for $k = 2$ and $n \geq 6$, we can also apply Theorem 3.30 (ii)).

The statement that if $2 \leq k \leq n-3$ (and thus $n \geq k+3 \geq 5$), then

$$f \text{ ext. quasiconvex} \xRightarrow{\neq} f \text{ ext. one convex}$$

follows from Theorem 3.34. ■

3.5 The ext convexity properties and the classical notions of convexity.

3.5.1 The projection maps

In this section we explore the relationship between the notions of ext. polyconvexity, ext. quasiconvexity and ext. one convexity and the classical notions of the calculus of variations namely rank one convexity, quasiconvexity and polyconvexity (see [25]). We first introduce some notations. As usual, by abuse of notations, we identify $\Lambda^k(\mathbb{R}^n)$ with $\mathbb{R}^{\binom{n}{k}}$.

Definition 3.39 (exterior projection) Let $2 \leq k \leq n$. To a matrix $\Xi \in \mathbb{R}^{\binom{n}{k-1} \times n}$, the upper indices being ordered alphabetically, written, depending on the context, as

$$\begin{aligned} \Xi &= \begin{pmatrix} \Xi_1^{1 \dots (k-1)} & \dots & \Xi_n^{1 \dots (k-1)} \\ \vdots & \ddots & \vdots \\ \Xi_1^{(n-k+2) \dots n} & \dots & \Xi_n^{(n-k+2) \dots n} \end{pmatrix} \\ &= (\Xi_i^I)_{i \in \{1, \dots, n\}}^{I \in \mathcal{T}^{k-1}} = \begin{pmatrix} \Xi_1^{1 \dots (k-1)} \\ \vdots \\ \Xi_n^{(n-k+2) \dots n} \end{pmatrix} = (\Xi_1, \dots, \Xi_n) \end{aligned}$$

we associate a map $\pi^{ext,k} : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \Lambda^k(\mathbb{R}^n)$ in the following way,

$$\pi^{ext,k}(\Xi) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k (-1)^{j+1} \Xi_{i_j}^{i_1 \dots i_{j-1} i_{j+1} \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{i=1}^n \Xi_i \wedge e^i$$

where

$$\Xi_i = \sum_{1 \leq i_1 < \dots < i_{k-1} \leq n} \Xi_i^{i_1 \dots i_{k-1}} e^{i_1} \wedge \dots \wedge e^{i_{k-1}} = \sum_{I \in \mathcal{T}^{k-1}} \Xi_i^I e^I.$$

Remark 3.40 Observe that this projection map can also be written as,²

$$\pi^{ext,k}(\Xi) = \sum_{I \in \mathcal{T}_k} \left(\sum_{j \in I} \text{sgn}(j, I_j) \Xi_j^{I_j} \right) e^I.$$

Remark 3.41 Note also that when $k = 2$, we find that $\pi^{ext,k} : \mathbb{R}^{n \times n} \rightarrow \Lambda^2(\mathbb{R}^n)$ is given by

$$\xi = \begin{pmatrix} \xi_1^1 & \dots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^n & \dots & \xi_n^n \end{pmatrix} = (\xi_1, \dots, \xi_n)$$

and

$$\pi^{ext,k}(\xi) = \sum_{i=1}^n \xi_i \wedge e^i = \sum_{1 \leq i < j \leq n} (\xi_j^i - \xi_i^j) e^i \wedge e^j$$

so that when restricted to the set of skew symmetric matrices, namely

$$\mathbb{R}_{as}^{n \times n} = \{\xi \in \mathbb{R}^{n \times n} : \xi^t = -\xi\}$$

we have

$$\pi^{ext,k}(\xi) = 2 \sum_{1 \leq i < j \leq n} \xi_j^i e^i \wedge e^j.$$

Similarly as above,

Definition 3.42 (interior projection) Let $1 \leq k \leq n-1$. To a matrix $\Xi \in \mathbb{R}^{\binom{n}{k+1} \times n}$, the upper indices being ordered alphabetically, written, depending on the context, as

$$\begin{aligned} \Xi &= \begin{pmatrix} \Xi_1^{1 \dots (k+1)} & \dots & \Xi_n^{1 \dots (k+1)} \\ \vdots & \ddots & \vdots \\ \Xi_1^{(n-k) \dots n} & \dots & \Xi_n^{(n-k) \dots n} \end{pmatrix} \\ &= (\Xi_i^I)_{i \in \{1, \dots, n\}}^{I \in \mathcal{T}_{k+1}^n} = \begin{pmatrix} \Xi_1^{1 \dots (k+1)} \\ \vdots \\ \Xi_n^{(n-k) \dots n} \end{pmatrix} = (\Xi_1, \dots, \Xi_n) \end{aligned}$$

²See Appendix A for the notation I_j .

we associate a map $\pi^{int,k} : \mathbb{R}^{\binom{n}{k+1} \times n} \rightarrow \Lambda^k(\mathbb{R}^n)$ in the following way,

$$\pi^{int,k}(\Xi) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i_{j-1} < \gamma < i_j} \Xi_{\gamma}^{i_1 \dots i_{j-1} i_{j+1} \dots i_k} \right) e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{i=1}^n \Xi_i \lrcorner e^i$$

where

$$\Xi_i = \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} \Xi_i^{i_1 \dots i_{k+1}} e^{i_1} \wedge \dots \wedge e^{i_{k+1}} = \sum_{I \in \mathcal{T}_{k+1}^n} \Xi_i^I e^I.$$

The following properties are easily obtained.

Proposition 3.43 *Let $2 \leq k \leq n$ and $\pi^{ext,k} : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \Lambda^k(\mathbb{R}^n)$ be as above.*

(i) *If $\alpha \in \Lambda^{k-1}(\mathbb{R}^n) \sim \mathbb{R}^{\binom{n}{k-1}}$ and $\beta \in \Lambda^1(\mathbb{R}^n) \sim \mathbb{R}^n$, then,*

$$\pi^{ext,k}(\alpha \otimes \beta) = \alpha \wedge \beta.$$

(ii) *Let $\omega \in C^1(\Omega; \Lambda^{k-1})$, then, by abuse of notations,*

$$\pi^{ext,k}(\nabla \omega) = d\omega.$$

Proof (i) We note that

$$\alpha \otimes \beta = \begin{pmatrix} \alpha^{1 \dots (k-1)} \beta_1 & \dots & \alpha^{1 \dots (k-1)} \beta_n \\ \vdots & \ddots & \vdots \\ \alpha^{(n-k+2) \dots n} \beta_1 & \dots & \alpha^{(n-k+2) \dots n} \beta_n \end{pmatrix}$$

so that

$$\pi^{ext,k}(\alpha \otimes \beta) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k (-1)^{j+1} \alpha^{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}} \beta_{i_j} e^{i_1} \wedge \dots \wedge e^{i_k} = \alpha \wedge \beta.$$

(ii) As above we have

$$\nabla \omega = \begin{pmatrix} \frac{\partial \omega^{1 \dots (k-1)}}{\partial x_1} & \dots & \frac{\partial \omega^{1 \dots (k-1)}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega^{(n-k+2) \dots n}}{\partial x_1} & \dots & \frac{\partial \omega^{(n-k+2) \dots n}}{\partial x_n} \end{pmatrix}$$

and thus $\pi^{ext,k}(\nabla \omega) = d\omega$ since

$$\pi^{ext,k}(\nabla \omega) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k (-1)^{j+1} \frac{\partial \omega^{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}}}{\partial x_{i_j}} e^{i_1} \wedge \dots \wedge e^{i_k}.$$

■

Similarly we have the following.

Proposition 3.44 *Let $1 \leq k \leq n-1$ and $\pi^{int,k} : \mathbb{R}^{\binom{n}{k+1} \times n} \rightarrow \Lambda^k(\mathbb{R}^n)$ be as above.*

(i) If $\alpha \in \Lambda^{k+1}(\mathbb{R}^n) \sim \mathbb{R}^{\binom{n}{k+1}}$ and $\beta \in \Lambda^1(\mathbb{R}^n) \sim \mathbb{R}^n$, then,

$$\pi^{\text{int},k}(\alpha \otimes \beta) = \alpha \lrcorner \beta.$$

(ii) Let $\omega \in C^1(\Omega; \Lambda^{k+1})$, then, by abuse of notations,

$$\pi^{\text{int},k}(\nabla \omega) = \delta \omega.$$

Proof (i) We note that

$$\alpha \otimes \beta = \begin{pmatrix} \alpha^{1 \cdots (k+1)} \beta_1 & \cdots & \alpha^{1 \cdots (k+1)} \beta_n \\ \vdots & \ddots & \vdots \\ \alpha^{(n-k) \cdots n} \beta_1 & \cdots & \alpha^{(n-k) \cdots n} \beta_n \end{pmatrix}$$

so that

$$\pi^{\text{int},k}(\alpha \otimes \beta) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i_{j-1} < \gamma < i_j} \alpha^{i_1 \cdots i_{j-1} \gamma i_{j+1} \cdots i_k} \beta_\gamma \right) e^{i_1} \wedge \cdots \wedge e^{i_k} = \alpha \lrcorner \beta.$$

(ii) As above we have

$$\nabla \omega = \begin{pmatrix} \frac{\partial \omega^{1 \cdots (k+1)}}{\partial x_1} & \cdots & \frac{\partial \omega^{1 \cdots (k+1)}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega^{(n-k) \cdots n}}{\partial x_1} & \cdots & \frac{\partial \omega^{(n-k) \cdots n}}{\partial x_n} \end{pmatrix}$$

and thus $\pi^{\text{int},k}(\nabla \omega) = \delta \omega$ since

$$\pi^{\text{int},k}(\nabla \omega) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i_{j-1} < \gamma < i_j} \frac{\partial \omega^{i_1 \cdots i_{j-1} \gamma i_{j+1} \cdots i_k}}{\partial x_\gamma} \right) e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

■

This is already enough to show the relation between rank one convexity and ext. one convexity and quasiconvexity and ext. quasiconvexity. But the relation between polyconvexity and ext. polyconvexity is much harder. We need an important formula (Proposition 3.45) and a crucial lemma (Lemma 3.47).

Proposition 3.45 (Adjugate formula) *If k is even, then for $2 \leq s \leq [n/k]$,*³

$$\left[\pi^{\text{ext},k}(\Xi) \right]^s = (s!) \sum_{I \in \mathcal{T}^{sk}} \left(\sum_{\substack{J=\{j_1 j_2 \dots j_s\}=[j_1 j_2 \dots j_s], \\ \tilde{I}=\{I^1 I^2 \dots I^s\}=[I^1, I^2, \dots, I^s] \\ J \cup \tilde{I}=I}} \text{sgn}(J; \tilde{I}) (\text{adj}_s \Xi)_{\tilde{I}}^J \right) e^I$$

³See Appendix A for the notation $\text{sgn}(J; \tilde{I})$.

and

$$\left[\pi^{\text{ext},k}(\Xi) \right]^s = 0 \quad \text{for } [n/k] < s \leq \min \left\{ n, \binom{n}{k-1} \right\}.$$

If k is odd,

$$\left[\pi^{\text{ext},k}(\Xi) \right]^s = 0 \quad \text{for all } s, \ 2 \leq s \leq \min \left\{ n, \binom{n}{k-1} \right\}.$$

Proof Except the first equality, everything else is trivial, by properties of the wedge power. So we prove the case when k is even and $2 \leq s \leq [n/k]$. We prove it by induction.

Step 1 To start the induction, we first prove the case when $s = 2$.

We have,

$$\pi^{\text{ext},k}(\Xi) = \sum_{I \in \mathcal{T}^k} \left(\sum_{j \in I} \text{sgn}(j, I_j) \Xi_j^{I_j} \right) e^I \quad (3.25)$$

So,

$$\begin{aligned} (\pi^{\text{ext},k}(\Xi))^2 &= \pi^{\text{ext},k}(\Xi) \wedge \pi^{\text{ext},k}(\Xi) \\ &= \sum_{I \in \mathcal{T}^{2k}} \left(\sum_{\substack{I^1, I^2 \\ I^1 \cup I^2 = I \\ I^1 \cap I^2 = \emptyset}} \text{sgn}(I^1, I^2) \left(\sum_{j_1 \in I^1} \text{sgn}(j_1, I_{j_1}^1) \Xi_{j_1}^{I_{j_1}^1} \right) \right. \\ &\quad \left. \left(\sum_{j_2 \in I^2} \text{sgn}(j_2, I_{j_2}^2) \Xi_{j_2}^{I_{j_2}^2} \right) \right) e^I \end{aligned} \quad (3.26)$$

Now, since k is even, we have

$$\text{sgn}(I^1, I^2) = \text{sgn}(I^2, I^1)$$

and hence,

$$\begin{aligned} &(\pi^{\text{ext},k}(\Xi))^2 \\ &= 2 \sum_{I \in \mathcal{T}^{2k}} \left(\sum_2^I \left(\text{sgn}([j_1, I_{j_1}^1], [j_2, I_{j_2}^2]) \text{sgn}(j_1, I_{j_1}^1) \text{sgn}(j_2, I_{j_2}^2) \Xi_{j_1}^{I_{j_1}^1} \Xi_{j_2}^{I_{j_2}^2} \right. \right. \\ &\quad \left. \left. + \text{sgn}([j_1, I_{j_2}^2], [j_2, I_{j_1}^1]) \text{sgn}(j_1, I_{j_2}^2) \text{sgn}(j_2, I_{j_1}^1) \Xi_{j_2}^{I_{j_1}^1} \Xi_{j_1}^{I_{j_2}^2} \right) \right) e^I \\ &= 2 \sum_{I \in \mathcal{T}^{2k}} \left(\sum_2^I (\text{sgn}(j_1, I_{j_1}^1, j_2, I_{j_2}^2) \Xi_{j_1}^{I_{j_1}^1} \Xi_{j_2}^{I_{j_2}^2} + \text{sgn}(j_1, I_{j_2}^2, j_2, I_{j_1}^1) \Xi_{j_2}^{I_{j_1}^1} \Xi_{j_1}^{I_{j_2}^2}) \right) e^I \end{aligned}$$

Now, since k is even,

$$\text{sgn}(j_1, I_{j_1}^1, j_2, I_{j_2}^2) = (-1)^{(k-1)} \text{sgn}(j_1, I_{j_2}^2, j_2, I_{j_1}^1) = -\text{sgn}(j_1, I_{j_2}^2, j_2, I_{j_1}^1)$$

Hence,

$$(\pi^{\text{ext},k}(\Xi))^2 = 2 \sum_{I \in \mathcal{T}_{2k}} \left(\sum_2^I \text{sgn}(j_1, I_{j_1}^1, j_2, I_{j_2}^2) (\Xi_{j_1}^{I_{j_1}^1} \Xi_{j_2}^{I_{j_2}^2} - \Xi_{j_2}^{I_{j_1}^1} \Xi_{j_1}^{I_{j_2}^2}) \right) e^I \quad (3.27)$$

That is,

$$(\pi^{\text{ext},k}(\Xi))^2 = 2 \sum_{I \in \mathcal{T}_{2k}} \left(\sum_2^I \text{sgn}(j_1, I_{j_1}^1, j_2, I_{j_2}^2) (\text{adj}_2 \Xi)_{j_1 j_2}^{I_{j_1}^1 I_{j_2}^2} \right) e^I \quad (3.28)$$

which proves the case for $s = 2$.

Step 2 We assume the result to be true for some $s \geq 2$ and show that it holds for $s + 1$, thus completing the induction. Now we know, by Laplace expansion for the determinants,

$$(\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} = \sum_{m=1}^{s+1} \Xi_{j_l}^{I^m} (-1)^{l+m} (\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \quad (3.29)$$

Hence,

$$(\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} = \frac{1}{s+1} \sum_{l=1}^{s+1} \sum_{m=1}^{s+1} \Xi_{j_l}^{I^m} (-1)^{l+m} (\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \quad (3.30)$$

Also,

$$\text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1}) = (-1)^{\{(l-1)+(m-1)(k-1)\}} \text{sgn}(j_l, I^m, \tilde{I}^{l,m}) \quad (3.31)$$

Here $\tilde{I}^{l,m}$ is a shorthand for the permutation $(\tilde{j}_1, \tilde{I}^1, \dots, \tilde{j}_s, \tilde{I}^s)$,

where

- $\tilde{j}_1 < \tilde{j}_2 < \dots < \tilde{j}_s$ and $\{\tilde{j}_1, \tilde{j}_2, \dots, \tilde{j}_s\} = \{j_1, j_2, \dots, \widehat{j_l}, \dots, j_{s+1}\}$
- $\tilde{I}^1 < \tilde{I}^2 < \dots < \tilde{I}^s$ and $\{\tilde{I}^1, \tilde{I}^2, \dots, \tilde{I}^s\} = \{I^1, I^2, \dots, \widehat{I^m}, \dots, I^{s+1}\}$.

Note that this means $\tilde{j}_r = j_r$ for $1 \leq r < l$ and $\tilde{j}_r = j_{r+1}$ for $l \leq r \leq s$. Similarly, $\tilde{I}^r = I^r$ for $1 \leq r < m$ and $\tilde{I}^r = I^{r+1}$ for $m \leq r \leq s$.

The easiest way to see (3.31) is to note that,

$$\begin{aligned} & \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1}) \\ &= (-1)^{\{(k-1)+2(k-1)+\dots+s(k-1)\}} \text{sgn}(j_1, j_2, \dots, j_{s+1}, I^1, I^2, \dots, I^{s+1}) \\ &= (-1)^{\frac{s(s+1)(k-1)}{2}} \text{sgn}(j_1, j_2, \dots, j_{s+1}, I^1, I^2, \dots, I^{s+1}) \\ &= (-1)^{\{(l-1)+(m-1)(k-1)\}} (-1)^{\frac{s(s+1)(k-1)}{2}} \text{sgn}(j_l, j_1, \dots, j_{s+1}, I^m, I^1, \dots, I^{s+1}) \\ &= (-1)^{\{(l-1)+(m-1)(k-1)\}} \text{sgn}(j_l, I^m, \tilde{I}^{l,m}) \end{aligned}$$

Now since k is even, $k - 1$ is odd and hence we have,⁴

$$(-1)^{\{(l-1)+(m-1)(k-1)\}} = (-1)^{l+m}.$$

So,

$$\begin{aligned} \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1}) &= (-1)^{l+m} \text{sgn}(j_l, I^m, \tilde{I}^{l,m}) \\ &= (-1)^{l+m} \text{sgn}(j_l, I^m) \text{sgn}(\tilde{I}^{l,m}) \text{sgn}([j_l, I^m], [\tilde{I}^{l,m}]) \end{aligned}$$

⁴See Appendix A for explanation of the notations.

Thus,

$$\begin{aligned} & \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1})(\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} \\ &= \frac{1}{(s+1)} \sum_{l,m=1}^{s+1} \text{sgn}([j_l, I^m], [\tilde{I}^{l,m}]) \text{sgn}(j_l, I^m) \Xi_{j_l}^{I^m} \text{sgn}(\tilde{I}^{lm})(\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \end{aligned}$$

Hence,⁵

$$\begin{aligned} & (s+1)! \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1})(\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} \right) e^I \\ &= \frac{(s+1)!}{(s+1)} \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \sum_{l,m=1}^{s+1} \text{sgn}([j_l, I^m], [\tilde{I}^{l,m}]) \text{sgn}(j_l, I^m) \Xi_{j_l}^{I^m} \right. \\ & \quad \left. \text{sgn}(\tilde{I}^{lm})(\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \right) e^I \\ &= (s!) \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \sum_{l,m=1}^{s+1} \text{sgn}([j_l, I^m], [\tilde{I}^{l,m}]) \text{sgn}(j_l, I^m) \Xi_{j_l}^{I^m} \right. \\ & \quad \left. \text{sgn}(\tilde{I}^{lm})(\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \right) e^I \end{aligned}$$

Now, rewriting the sum, we obtain,

$$\begin{aligned} & (s!) \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \sum_{l,m=1}^{s+1} \text{sgn}([j_l, I^m], [\tilde{I}^{l,m}]) \text{sgn}(j_l, I^m) \Xi_{j_l}^{I^m} \right. \\ & \quad \left. \text{sgn}(\tilde{I}^{lm})(\text{adj}_s \Xi)_{j_1 \dots \widehat{j_l} \dots j_{s+1}}^{I^1 \dots \widehat{I^m} \dots I^{s+1}} \right) e^I \\ &= \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{\substack{I' \subset I \\ I' \in \mathcal{T}_k}} \left(\text{sgn}(I', [I \setminus I']) \left(\sum_{j \in I'} \text{sgn}(j, I'_j) \Xi_j^{I'_j} \right) \right. \right. \\ & \quad \left. \left. \left(s! \left(\sum_s^{[I \setminus I']} \text{sgn}(\tilde{j}_1, \tilde{I}^1, \dots, \tilde{j}_s, \tilde{I}^s)(\text{adj}_s \Xi)_{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_s}^{\tilde{I}^1 \tilde{I}^2 \dots \tilde{I}^s} \right) \right) \right) \right) e^I \end{aligned}$$

To see that the RHS of the above equation is indeed just a rewriting of the LHS, note that once we have expanded all the sums on both sides, the map sending $j_l \mapsto j$, $I^m \mapsto I'_j$, $I^1, \dots, \widehat{I^m}, \dots, I^{s+1}$ to $\tilde{I}^1, \tilde{I}^2, \dots, \tilde{I}^s$ respectively and $j_1, \dots, \widehat{j_l}, \dots, j_{s+1}$ to $\tilde{j}_1, \tilde{j}_2, \dots, \tilde{j}_s$ respectively is a bijection between the terms on the two sides of the equation.

⁵See Appendix A for explanation of the notations.

So, we have,

$$\begin{aligned}
& (s+1)! \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1}) (\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} \right) e^I \\
&= \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{\substack{I' \subset I \\ I' \in \mathcal{T}_k}} \text{sgn}(I', [I \setminus I']) \left(\text{coefficient of } e^{I'} \text{ in } \pi^{\text{ext}, k}(\Xi) \right) \right) \\
&\quad \times \left(\text{coefficient of } e^{[I \setminus I']} \text{ in } \left[\pi^{\text{ext}, k}(\Xi) \right]^s \right) e^I, \tag{3.32}
\end{aligned}$$

by the induction hypothesis.

But this implies,

$$\begin{aligned}
& (s+1)! \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\sum_{s+1}^I \text{sgn}(j_1, I^1, \dots, j_{s+1}, I^{s+1}) (\text{adj}_{s+1} \Xi)_{j_1 j_2 \dots j_{s+1}}^{I^1 I^2 \dots I^{s+1}} \right) e^I \\
&= \sum_{I \in \mathcal{T}^{(s+1)k}} \left(\text{coefficient of } e^I \text{ in } \left[\pi^{\text{ext}, k}(\Xi) \right]^{s+1} \right) e^I = \left[\pi^{\text{ext}, k}(\Xi) \right]^{s+1}, \tag{3.33}
\end{aligned}$$

completing the induction and thereby proving the desired result. ■

Since we have seen that $[\pi^{\text{ext}, k}(\Xi)]^s$ depends only on $\text{adj}_s \Xi$, we are now in a position to define a linear projection for every value of s . These maps will be useful later.

Notation 3.46 For every value of $2 \leq s \leq \min \left\{ n, \binom{n}{k-1} \right\}$, we define the linear projection maps $\pi_s^{\text{ext}, k} : \mathbb{R}^{\left(\binom{n}{k-1} \right) \times \binom{n}{s}} \rightarrow \Lambda^{ks}(\mathbb{R}^n)$ by the condition,

$$\pi_s^{\text{ext}, k}(\text{adj}_s(\Xi)) = \left[\pi^{\text{ext}, k}(\Xi) \right]^s \text{ for all } \Xi \in \mathbb{R}^{\binom{n}{k-1} \times n}.$$

It is clear that this condition uniquely defines the projection maps. For the sake of consistency, we define, $\pi_1^{\text{ext}, k} = \pi^{\text{ext}, k}$ and $\pi_0^{\text{ext}, k}$ is defined to be the identity map from \mathbb{R} to \mathbb{R} .

3.5.2 A crucial lemma

Now, to show the relation between polyconvexity and ext. polyconvexity, we need a lemma.

Lemma 3.47 Let $N = \binom{n}{k-1}$. Let

$$g(X, d) = f(\pi^{\text{ext}, k}(X)) - \sum_{s=0}^{\min\{N, n\}} \langle d_s, \text{adj}_s X \rangle$$

where $d = (d_1, d_2, \dots, d_{\min\{N, n\}})$, $d_s \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}$ for all $0 \leq s \leq \min\{N, n\}$ and $X \in \mathbb{R}^{N \times n}$. If for a given vector d , the function $X \mapsto g(X, d)$ achieves a minimum over $\mathbb{R}^{N \times n}$, then for all $0 \leq s \leq \min\{N, n\}$,

$$\langle d_s, Y \rangle = \langle \pi_s^{\text{ext}, k}(d_s), \pi_s^{\text{ext}, k}(Y) \rangle \text{ for all } Y \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}.$$

The lemma is quite technical and quite heavy in terms of notations. So before proceeding to prove the lemma as stated, it might be helpful to spell out the idea of the proof. The plan is always the same. In short, if $d_s \not\cong \pi_s^{\text{ext},k}(d_s)$ for any $0 \leq s \leq \min\{N, n\}$, then we can always choose a matrix X such that $g(X, d)$ can be made to be smaller than any given real number, contradicting the hypothesis that the map $X \mapsto g(X, d)$ assumes a finite minimum. Note that since f takes values in \mathbb{R} , i.e finite values, if $X \mapsto g(X, d)$ achieves a minimum, the minimum must be finite.

We shall show the lemma in three cases. The first one, the case for $k = 2$ is mostly for the sake of illustration. The other two being the case of k being an even integer ($k > 2$) and the case of k being an odd integer.

Example Case : $k = 2$, n arbitrary

Proof Fix a vector d and assume that for this d , the function $X \mapsto g(X, d)$ achieves a minimum over $\mathbb{R}^{N \times n}$. Note that the minimum is a finite real number (since f is finite).

Step 1 We will first show that all adjugates with a common index must have zero coefficients. More precisely, we claim,

Claim 3.48 For every $1 \leq s \leq \min\{N, n\}$, for every $J, I \in \mathcal{T}^s$,

$$(d_s)_J^I = 0 \text{ whenever } I \cap J \neq \emptyset. \quad (3.34)$$

Step 1a We prove claim 3.48, using induction over s . To start the induction, we first show the case $s = 1$. We choose $X = \lambda e^i \otimes e^i$, then clearly $\pi^{\text{ext},2}(X) = 0$. Also, $g(X, d) = f(0) - \lambda (d_1)_i^i$. By letting λ to $+\infty$ and $-\infty$ respectively, we deduce that $(d_1)_i^i = 0$, since otherwise we obtain a contradiction to the fact that g achieves a finite minima.

Step 1b Now we assume that claim 3.48 holds for all $1 \leq s \leq p$ and prove the result for $s = p + 1$.

We consider $(d_s)_{j_1 j_2 \dots j_{p+1}}^{i_1 i_2 \dots i_{p+1}}$ with $i_l = j_m$ for some $1 \leq l, m \leq p + 1$.

Now we first order the rest of the indices (other than the common index) in subscripts and superscripts. Let $\tilde{i}_1 < \tilde{i}_2 < \dots < \tilde{i}_p$ and $\tilde{j}_1 < \tilde{j}_2 < \dots < \tilde{j}_p$ represent the indices in the set $\{i_1, i_2, \dots, i_{p+1}\} \setminus \{i_l\}$ and $\{j_1, j_2, \dots, j_{p+1}\} \setminus \{j_m\}$ respectively.

Now we choose,

$$X = \lambda e^{i_l} \otimes e^{j_m} + \sum_{r=1}^p e^{\tilde{i}_r} \otimes e^{\tilde{j}_r}.$$

Since $i_l = j_m$, we get $\pi^{\text{ext},k}(X)$ is independent of λ . Also, all lower order non-constant adjugate of X must contain the index $i_l = j_m$ both in subscript and in superscript and hence their coefficients are 0 by the induction hypothesis. Hence, the only non-constant adjugate of X appearing in the expression for $g(X, d)$ is ,

$$(\text{adj}_{p+1} X)_{j_1 j_2 \dots j_{p+1}}^{i_1 i_2 \dots i_{p+1}} = (-1)^\alpha \lambda,$$

where α is a fixed integer. Since whether α is odd or even has no bearing on our following argument, we would not bother ourselves with it. Now,

$$g(X, d) = (-1)^\alpha \lambda (d_s)_{j_1 j_2 \dots j_{p+1}}^{i_1 i_2 \dots i_{p+1}} + \text{constants} .$$

Again as in Step 1a, we let λ to $+\infty$ and $-\infty$ and we deduce, by the same argument, $(d_s)_{j_1 j_2 \dots j_{p+1}}^{i_1 i_2 \dots i_{p+1}} = 0$. This completes the induction and proves the claim.

Step 2 By Step 1, it is clear that $d_s = 0$ for all $s > [\frac{n}{2}]$, since in all those cases, there must be a common index. Now we will show that the coefficients of two different adjugates having the same set of indices are related in a precise manner. More precisely, we claim,

Claim 3.49 *For every $1 \leq s \leq [\frac{n}{2}]$,*

$$\text{sgn}(J; I) (d_s)_J^I = \text{sgn}(\tilde{J}; \tilde{I}) (d_s)_{\tilde{J}}^{\tilde{I}} . \quad (3.35)$$

whenever $[J, I] = [\tilde{J}, \tilde{I}]$, with $J, I, \tilde{J}, \tilde{I} \in \mathcal{T}^s$ and $J \cap I = \emptyset$.

Step 2a We will prove the claim by induction over s . To start the induction, we first prove it for the case $s = 1$.

For the case $s = 1$, we just need to prove,

$$\text{sgn}(j, i) (d_1)_j^i = \text{sgn}(i, j) (d_1)_i^j . \quad (3.36)$$

We choose $X = \lambda e^j \otimes e^i + \lambda e^i \otimes e^j$. Clearly, $\pi^{\text{ext}, 2}(X) = 0$ and this gives,

$$g(X, d) = f(0) + \lambda \left((d_1)_j^i + (d_1)_i^j \right) ,$$

where we have used Step 1 to deduce that $(d_2)_{ij}^{ij} = 0$ (assuming $i < j$). Letting λ to $+\infty$ and $-\infty$, we get (3.36).

Step 2b Now we assume the result for all $1 \leq s \leq s_0$ and show it for $s = s_0 + 1$. Take $J = \{j_1 j_2 \dots j_{s_0+1}\}$, $I = \{i_1 i_2 \dots i_{s_0+1}\}$ and $\tilde{J} = \{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_{s_0+1}\}$ and $\tilde{I} = \{\tilde{i}_1 \tilde{i}_2 \dots \tilde{i}_{s_0+1}\}$. Now since we have $[J, I] = [\tilde{J}, \tilde{I}]$, the strings (J, I) and (\tilde{J}, \tilde{I}) are permutations of each other, preserving an order relation. The order relation is easy to write down. $j_1 < j_2 < \dots < j_{s_0+1}$ and $i_1 < i_2 < \dots < i_{s_0+1}$. Thus the two above mentioned strings can be related by any permutation (of $2(s_0 + 1)$ indices) that respects this order. Since any such permutation can be factorized into a product of 1-flips (see Appendix A for definition), it is enough to prove the claim in case of a 1-flip.

We now assume (J, I) and (\tilde{J}, \tilde{I}) are related by a 1-flip interchanging the subscript $j_l \in J$ with superscript $i_m \in I$ and keep all the other indices unchanged. Also, we assume that after the interchange, the new position of the index j_l in the superscript is p and the new position of the index i_m in the subscript is q , i.e ,

$$j_l = \tilde{i}_p \quad ; \quad i_m = \tilde{j}_q . \quad (3.37)$$

We also order the remaining indices and assume

$$\{\hat{J}\} = \{\hat{j}_1 \hat{j}_2 \dots \hat{j}_{s_0}\} = [J \setminus \{j_l\}] \quad \text{and} \quad \{\hat{I}\} = \{\hat{i}_1 \hat{i}_2 \dots \hat{i}_{s_0}\} = [I \setminus \{i_m\}].$$

Now we choose,

$$X = \lambda e^{j_l} \otimes e^{i_m} + \lambda e^{\tilde{j}_q} \otimes e^{\tilde{i}_p} + \sum_{1 \leq r \leq s_0} e^{\hat{j}_r} \otimes e^{\hat{i}_r}. \quad (3.38)$$

Note that $\pi^{\text{ext},2}(X)$ is independent of λ , by (3.37). Also, all non-constant adjugates of X appearing with possibly non-zero coefficients in the expression for $g(X, d)$ have, either j_l in subscript and i_m in superscript or has \tilde{j}_q as a subscript and \tilde{i}_p as a superscript, but never both as then they have zero coefficients by Step 1. Also, these adjugates occur in pairs. More precisely, for every non-constant adjugates of X appearing with possibly non-zero coefficients in the expression for $g(X, d)$ having j_l in subscript and i_m in superscript, there is a non-constant adjugates of X appearing with possibly non-zero coefficients in the expression for $g(X, d)$ having \tilde{j}_q as a subscript and \tilde{i}_p as a superscript. We will make this last statement more precise shortly.

Step 2c Now first we show that, for any $1 \leq s \leq s_0 + 1$, for any subset $\{\bar{j}_1, \bar{j}_2, \dots, \bar{j}_{s-1}\} = \{\bar{J}_{s-1}\} \subset \{\hat{J}\}$, and any subset $\{\bar{i}_1, \bar{i}_2, \dots, \bar{i}_{s-1}\} = \{\bar{I}_{s-1}\} \subset \{\hat{I}\}$, we have,

$$\frac{(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]}^{[i_m \bar{I}_{s-1}]}}{\text{sgn}([j_l \bar{J}_{s-1}]; [i_m \bar{I}_{s-1}])} = - \frac{(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{i}_p \bar{I}_{s-1}]}}{\text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{i}_p \bar{I}_{s-1}])}. \quad (3.39)$$

Let a_1 be the position of j_l in $[j_l \bar{J}_{s-1}]$, b_1 be the position of i_m in $[i_m \bar{I}_{s-1}]$, a_2 be the position of \tilde{j}_q in $[\tilde{j}_q \bar{J}_{s-1}]$ and b_2 be the position of \tilde{i}_p in $[\tilde{i}_p \bar{I}_{s-1}]$.

Then we have,

$$\begin{aligned} \text{sgn}([j_l \bar{J}_{s-1}]; [i_m \bar{I}_{s-1}]) &= (-1)^{\{(a_1-1)+(b_1-1)\}} \text{sgn}(j_l, i_m) \text{sgn}(\bar{J}_{s-1}; \bar{I}_{s-1}) \\ &\quad \text{sgn}([j_l, i_m], [(\bar{J}_{s-1}; \bar{I}_{s-1})]) \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} \text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{i}_p \bar{I}_{s-1}]) &= (-1)^{\{(a_2-1)+(b_2-1)\}} \text{sgn}(\tilde{j}_q, \tilde{i}_p) \text{sgn}(\bar{J}_{s-1}; \bar{I}_{s-1}) \\ &\quad \text{sgn}([\tilde{j}_q, \tilde{i}_p], [(\bar{J}_{s-1}; \bar{I}_{s-1})]). \end{aligned} \quad (3.41)$$

We also have,

$$(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]}^{[i_m \bar{I}_{s-1}]} = (-1)^{a_1+b_1} \lambda (\text{adj}_{s-1} X)_{[\bar{J}_{s-1}]}^{[\bar{I}_{s-1}]}. \quad (3.42)$$

and

$$(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{i}_p \bar{I}_{s-1}]} = (-1)^{a_2+b_2} \lambda (\text{adj}_{s-1} X)_{[\bar{J}_{s-1}]}^{[\bar{I}_{s-1}]}. \quad (3.43)$$

Now since $[j_l, i_m] = [\tilde{j}_q, \tilde{i}_p]$ and $\text{sgn}(j_l, i_m) = -\text{sgn}(\tilde{j}_q, \tilde{i}_p)$ we have,

$$\frac{\text{sgn}([j_l \bar{J}_{s-1}]; [i_m \bar{I}_{s-1}])}{\text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{i}_p \bar{I}_{s-1}])} = -(-1)^{(a_1+b_1)-(a_2+b_2)}. \quad (3.44)$$

Also, clearly,

$$\frac{(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]^{[i_m \bar{I}_{s-1}]}}{(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]^{[\tilde{i}_p \bar{I}_{s-1}]}}} = (-1)^{(a_1+b_1)-(a_2+b_2)}. \quad (3.45)$$

Combining the two equations above, the result follows.

Step 2d We now finish the proof of claim (3.49). By Step 2c, we have,

$$\begin{aligned} g(X, d) = & \lambda \left\{ (-1)^\alpha \left(\text{sgn}(J; I) (d_{s_0+1})_J^I - \text{sgn}(\tilde{J}; \tilde{I}) (d_{s_0+1})_{\tilde{J}}^{\tilde{I}} \right) \right. \\ & + \sum_{s=1}^{s_0} \frac{1}{(s-1)!} \sum_{\substack{\bar{I}_{s-1} \subset \hat{I} \\ \bar{J}_{s-1} \subset \hat{J}}} (-1)^{\beta'_s} \left(\begin{aligned} & \text{sgn}([j_l \bar{J}_{s-1}]; [i_m \bar{I}_{s-1}]) (d_s)_{[j_l \bar{J}_{s-1}]}^{[i_m \bar{I}_{s-1}]} \\ & - \text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{i}_p \bar{I}_{s-1}]) (d_s)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{i}_p \bar{I}_{s-1}]} \end{aligned} \right) \Bigg\} \\ & + \text{constants} . \end{aligned}$$

By the induction hypothesis, the sum inside the braces in the above expression is 0. Hence, we obtain,

$$g(X, d) = (-1)^\alpha \lambda \left(\text{sgn}(J; I) (d_{s_0+1})_J^I - \text{sgn}(\tilde{J}; \tilde{I}) (d_{s_0+1})_{\tilde{J}}^{\tilde{I}} \right) + \text{constants} . \quad (3.46)$$

Letting λ to $+\infty$ and $-\infty$, we obtain the claim.

Step 3 By proposition (3.45), the claims (3.48) and (3.49) imply the result and finishes the proof. ■

Now we prove the lemma in complete generality.

Proof Let us fix a vector d and assume that for this d , the function $X \mapsto g(X, d)$ achieves a minimum over $\mathbb{R}^{N \times n}$.

We will first show that all adjugates with a common index between subscripts and superscripts must have zero coefficients. More precisely, we claim that,

Claim 3.50 *For any $2 \leq k \leq n$ and for every $1 \leq s \leq \min\{N, n\}$, for every $J \in \mathcal{T}^s, I = \{I^1 \dots I^s\}$ where $I^1, \dots, I^s \in \mathcal{T}^{k-1}$, we have,*

$$(d_s)_J^I = 0 \text{ whenever } I \cap J \neq \emptyset.$$

We prove claim 3.50, using induction over s . To start the induction, we first show the case $s = 1$. Let $j \in I$, where $I \in \mathcal{T}^{k-1}$. We choose $X = \lambda e^j \otimes e^I$, then clearly $\pi^{\text{ext}, 2}(X) = 0$. Also, $g(X, d) = f(0) - \lambda (d_1)_j^I$. By letting λ to $+\infty$ and $-\infty$ respectively, we deduce that $(d_1)_j^I = 0$, since otherwise we obtain a contradiction to the fact that g achieves a finite minima.

Now we assume that claim 3.50 holds for all $1 \leq s \leq p$ and prove the result for $s = p + 1$. We consider $(d_{p+1})_{j_1 \dots j_{p+1}}^{I^1 \dots I^{p+1}}$ with $j_l \in I^m$ for some $1 \leq l, m \leq p + 1$.

Now we first order the rest of the indices (other than the common index) in subscripts and the rest of the multiindices (other than the one with the common index) in superscripts. Let $\tilde{I}^1 < \dots < \tilde{I}^p$ and $\tilde{j}_1 < \dots < \tilde{j}_p$ represent the multiindices and indices in the sets $\{I^1, \dots, I^{p+1}\} \setminus \{I^m\}$ and $\{j_1, \dots, j_{p+1}\} \setminus \{j_l\}$ respectively.

Now we choose,

$$X = \lambda e^{j_l} \otimes e^{I^m} + \sum_{r=1}^p e^{\tilde{j}_r} \otimes e^{\tilde{I}^r}.$$

Since $j_l \in I^m$, we get $\pi(X)$ is independent of λ . Also, all lower order non-constant adjugates of X must contain the index j_l both in subscript and in superscript and hence their coefficients are 0 by the induction hypothesis. Hence, the only non-constant adjugate of X appearing in the expression for $g(X, d)$ is,

$$(\text{adj}_{p+1} X)_{j_1 \dots j_{p+1}}^{I^1 \dots I^{p+1}} = (-1)^\alpha \lambda,$$

where α is a fixed integer. Now,

$$g(X, d) = (-1)^{\alpha+1} \lambda (d_{p+1})_{j_1 \dots j_{p+1}}^{I^1 \dots I^{p+1}} + \text{constants}.$$

Again as before, we let λ to $+\infty$ and $-\infty$ and we deduce, by the same argument, $(d_{p+1})_{j_1 \dots j_{p+1}}^{I^1 \dots I^{p+1}} = 0$. This completes the induction and proves the claim.

At this point we split the proof in two cases, the case when k is an even integer and the case when k is an odd integer.

Case 1: k is even

Note that, unless $k = 2$, it does not follow from above that $d_s = 0$ for all $s \geq [\frac{n}{k}]$. The possibility that two different blocks of multiindices in the superscript have some index in common has not been ruled out. Now we will show that the coefficients of two different adjugates having the same set of indices are related in the following way:

Claim 3.51 *For every $s \geq 1$,*

$$\text{sgn}(J; I) (d_s)_J^I = \text{sgn}(\tilde{J}; \tilde{I}) (d_s)_{\tilde{J}}^{\tilde{I}},$$

whenever $J \cup I = \tilde{J} \cup \tilde{I}$, with $J, \tilde{J} \in \mathcal{T}^s$, $I = \{I^1 \dots I^s\} = [I^1, \dots, I^s]$, $\tilde{I} = \{\tilde{I}^1 \dots \tilde{I}^s\} = [\tilde{I}^1, \dots, \tilde{I}^s]$, $I^1, \dots, I^s, \tilde{I}^1, \dots, \tilde{I}^s \in \mathcal{T}^{k-1}$ and $J \cap I = \emptyset$. In particular, given any $U \in \mathcal{T}^{ks}$, there exists a constant $\mathcal{D}_U \in \mathbb{R}$ such that,

$$\text{sgn}(J; I) (d_s)_J^I = \mathcal{D}_U, \tag{3.47}$$

for all $J \cup I = U$ with $J \in \mathcal{T}^s$, $I = \{I^1 \dots I^s\} = [I^1, \dots, I^s]$, $I^1, \dots, I^s \in \mathcal{T}^{k-1}$.

We will prove the claim again by induction over s . We first prove it for the case $s = 1$.

For the case $s = 1$, we just need to prove, for any index j , any multiindex $I \in \mathcal{T}^{k-1}$ such that $j \cap I = \emptyset$, we have

$$\text{sgn}(j, I) (d_1)_j^I = \text{sgn}(\tilde{j}, \tilde{I}) (d_1)_{\tilde{j}}^{\tilde{I}}, \quad (3.48)$$

where $[j, I] = [\tilde{j}, \tilde{I}]$. We choose $X = \lambda \text{sgn}(j, I) e^j \otimes e^I - \lambda \text{sgn}(\tilde{j}, \tilde{I}) e^{\tilde{j}} \otimes e^{\tilde{I}}$. Clearly, $\pi(X) = 0$ and this gives,

$$g(X, d) = f(0) + \lambda \left(\text{sgn}(j, I) (d_1)_j^I - \text{sgn}(\tilde{j}, \tilde{I}) (d_1)_{\tilde{j}}^{\tilde{I}} \right),$$

where we have used claim 3.50 to deduce that $(d_2)_{[j\tilde{j}]}^{[I\tilde{I}]} = 0$. Letting λ to $+\infty$ and $-\infty$, we get (3.48).

Now we assume the result for all $1 \leq s \leq s_0$ and show it for $s = s_0 + 1$. Suppose first $[I^1 \dots I^{s_0+1} j_1 \dots j_{s_0+1}] = [\tilde{I}^1 \dots \tilde{I}^{s_0+1} \tilde{j}_1 \dots \tilde{j}_{s_0+1}]$. Note that the sets $\{I^1 \dots I^{s_0+1} j_1 \dots j_{s_0+1}\}$ and $\{\tilde{I}^1 \dots \tilde{I}^{s_0+1} \tilde{j}_1 \dots \tilde{j}_{s_0+1}\}$ are permutations of each other, preserving an order relation given by $j_1 < \dots < j_{s_0+1}$, $\tilde{j}_1 < \dots < \tilde{j}_{s_0+1}$, $I^1 < \dots < I^{s_0+1}$ and $\tilde{I}^1 < \dots < \tilde{I}^{s_0+1}$. Thus the aforementioned sets can be related by any permutation (of $k(s_0 + 1)$ indices) that respects this order. Since any such permutation is a product of k -flips, it is enough to prove the claim in case of k -flips, cf. definition A.3.

We now assume (J, I) and (\tilde{J}, \tilde{I}) are related by a k -flip interchanging the subscript j_l with one index in the superscript block I^m and keep all the other indices unchanged. Also, we assume that after the interchange, the position of the multiindex containing j_l in the superscript is p and the new position of the index from the multiindex I^m in the subscript is q , i.e. $j_l \in \tilde{I}^p$ and $\tilde{j}_q \in I^m$. We also order the remaining indices and assume ,

$$\check{I} = [\check{I}^1, \dots, \check{I}^{s_0}] = \{\check{I}^1 \dots \check{I}^{s_0}\} = \left\{ I^1 \dots \widehat{I^m} \dots I^{s_0+1} \right\},$$

and

$$\check{J} = [\check{j}_1 \dots \check{j}_{s_0}] = \{\check{j}_1 \dots \check{j}_{s_0}\} = \left\{ j_1 \dots \widehat{j_l} \dots j_{s_0+1} \right\}$$

respectively. Now we choose,

$$X = \lambda \text{sgn}(j_l, I^m) e^{j_l} \otimes e^{I^m} - \lambda \text{sgn}(\tilde{j}_q, \tilde{I}^p) e^{\tilde{j}_q} \otimes e^{\tilde{I}^p} + \sum_{1 \leq r \leq s_0} e^{\check{j}_r} \otimes e^{\check{I}^r}.$$

Note that $\pi^{\text{ext}, k}(X)$ is independent of λ . Also, all non-constant adjugates of X appearing with possibly non-zero coefficients in the expression for $g(X, d)$ have, either j_l in subscript and I^m in superscript or has \tilde{j}_q as a subscript and \tilde{I}^p as a superscript, but never both as then they have zero coefficients by claim 3.50. Also, these adjugates occur in pairs. More precisely, for every non-constant adjugate of X appearing with possibly non-zero coefficients in the expression for $g(X, d)$ having j_l in subscript and I^m in superscript, there is one having \tilde{j}_q in subscript and \tilde{I}^p in superscript.

Let us show that, for any $1 \leq s \leq s_0 + 1$, any subset $\bar{J}_{s-1} = \{\bar{j}_1, \dots, \bar{j}_{s-1}\} \subset \check{J}$ of s indices and any choice of $s - 1$ multiindices $\bar{I}^1, \dots, \bar{I}^{s-1}$ out of s_0 multiindices $\check{I}^1, \dots, \check{I}^{s_0}$, we have,

$$\frac{(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]}^{[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]}}{\text{sgn}([j_l \bar{J}_{s-1}]; [I^m, \bar{I}^1, \dots, \bar{I}^{s-1}])} = - \frac{(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]}}{\text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}])}. \quad (3.49)$$

Let a_1 be the position of j_l in $[j_l \bar{J}_{s-1}]$, a_2 be the position of \tilde{j}_q in $[\tilde{j}_q \bar{J}_{s-1}]$, b_1 be the position of I^m in $[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]$ and b_2 be the position of \tilde{I}^p in $[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]$.

Since k is even,

$$\begin{aligned} & \text{sgn}([j_l \bar{J}_{s-1}]; [I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]) \\ &= (-1)^{\{(a_1-1)+(b_1-1)\}} \text{sgn}(j_l, I^m) \text{sgn}(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\}) \\ & \quad \text{sgn}([j_l, I^m], [(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\})]), \end{aligned}$$

and

$$\begin{aligned} & \text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]) \\ &= (-1)^{\{(a_2-1)+(b_2-1)\}} \text{sgn}(\tilde{j}_q, \tilde{I}^p) \text{sgn}(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\}) \\ & \quad \text{sgn}([\tilde{j}_q, \tilde{I}^p], [(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\})]). \end{aligned}$$

We also have,

$$(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]}^{[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]} = (-1)^{a_1+b_1} \text{sgn}(j_l, I^m) \lambda (\text{adj}_{s-1} X)_{[\bar{J}_{s-1}]}^{[\bar{I}^1, \dots, \bar{I}^{s-1}]},$$

and

$$(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]} = -(-1)^{a_2+b_2} \text{sgn}(\tilde{j}_q, \tilde{I}^p) \lambda (\text{adj}_{s-1} X)_{[\bar{J}_{s-1}]}^{[\bar{I}^1, \dots, \bar{I}^{s-1}]}.$$

Combining the four equations above, the result follows.

We now finish the proof of claim 3.51. Using (3.49), we have,

$$\begin{aligned} g(X, d) &= \lambda \left\{ (-1)^\alpha \left(\text{sgn}(J; I) (d_{s_0+1})_J^I - \text{sgn}(\tilde{J}; \tilde{I}) (d_{s_0+1})_{\tilde{J}}^{\tilde{I}} \right) \right. \\ & \quad \left. + \sum_{s=1}^{s_0} \sum^s k_{s,\gamma} \begin{pmatrix} \text{sgn}([j_l \bar{J}_{s-1}]; [I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]) (d_s)_{[j_l \bar{J}_{s-1}]}^{[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]} \\ - \text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]) (d_s)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]} \end{pmatrix} \right\} \\ & \quad + \text{constants}, \end{aligned}$$

where \sum^s is a shorthand, for every $1 \leq s \leq s_0$, for the sum over all possible such choices of $\bar{J}_{s-1}, \bar{I}^1, \bar{I}^2, \dots, \bar{I}^{s-1}$ and $k_{s,\gamma}$ is a generic placeholder for the constants appearing before each term of the sum and α is an integer.

By the induction hypothesis, the sum on the right hand side of the above expression is 0. Hence, we obtain,

$$g(X, d) = (-1)^\alpha \lambda \left(\text{sgn}(J; I) (d_{s_0+1})_J^I - \text{sgn}(\tilde{J}; \tilde{I}) (d_{s_0+1})_{\tilde{J}}^{\tilde{I}} \right) + \text{constants}.$$

Letting λ to $+\infty$ and $-\infty$, the claim is proved by induction.

Note that by virtue of claim 3.51, claim 3.50 now implies, that for every $1 \leq s \leq \min\{N, n\}$, for every $J \in \mathcal{T}^s$, $I = \{I^1 \dots I^s\}$ where $I^1, \dots, I^s \in \mathcal{T}^{k-1}$, we have,

$$(d_s)_J^I = 0 \text{ whenever either } I \cap J \neq \emptyset \text{ or } I^l \cap I^m \neq \emptyset \text{ for some } 1 \leq l < m \leq s. \quad (3.50)$$

Indeed, if $I \cap J \neq \emptyset$, we are done, using claim 3.50. So let us assume $I \cap J = \emptyset$ but $I^l \cap I^m \neq \emptyset$ for some $1 \leq l < m \leq s$. Then there exists an index i such that $i \in I^l$ and $i \in I^m$, we consider the k -flip interchanging some index j from subscript with the index i in I^l . More precisely, let $\tilde{J} \in \mathcal{T}^s$ and $\tilde{I}^l \in \mathcal{T}^{k-1}$ be such that $i \in \tilde{J}$, $\tilde{J} \setminus \{i\} \subset J$, $I^l \setminus \{i\} \subset \tilde{I}^l$ and $J \cup I^l = \tilde{J} \cup \tilde{I}^l$, then by claim 3.51 we have,

$$\text{sgn}(J; I) (d_s)_J^I = \text{sgn} \left(\tilde{J}; \left[\tilde{I}^l, I^1, \dots, \hat{I}^l, \dots, I^s \right] \right) (d_s)_{\tilde{J}}^{\left[\tilde{I}^l, I^1, \dots, \hat{I}^l, \dots, I^s \right]}.$$

Since, $i \in \tilde{J}$ and $i \in I^m$, $\tilde{J} \cap \left[\tilde{I}^l, I^1, \dots, \hat{I}^l, \dots, I^s \right] \neq \emptyset$, the right hand side of above equation is 0 and so $(d_s)_J^I = 0$, which proves (3.50). So this now implies, $d_s = 0$ for all $s \geq \lfloor \frac{n}{k} \rfloor$. Hence we have, using (3.47), (3.50) and proposition 3.45,

$$\begin{aligned} \langle d_s, \text{adj}_s Y \rangle &= \sum_{I \in \mathcal{T}^{sk}} \sum_s^I (d_s)_J^{\tilde{I}} (\text{adj}_s Y)_J^{\tilde{I}} \\ &= \sum_{I \in \mathcal{T}^{sk}} \sum_s^I \text{sgn}(J; \tilde{I}) (d_s)_J^{\tilde{I}} \text{sgn}(J; \tilde{I}) (\text{adj}_s Y)_J^{\tilde{I}} \\ &= \sum_{I \in \mathcal{T}^{sk}} \frac{1}{s!} \mathcal{D}_I \sum_s^I (s!) \text{sgn}(J; \tilde{I}) (\text{adj}_s Y)_J^{\tilde{I}} \\ &= \langle \mathcal{D}_s, \pi_s^{\text{ext}, k}(\text{adj}_s Y) \rangle, \end{aligned}$$

where $\mathcal{D}_s = \frac{1}{s!} \sum_{I \in \mathcal{T}^{sk}} \mathcal{D}_I e^I$, which finishes the proof when k is even.

Case 3: k is odd

In this case, by proposition 3.45, it is enough to show that all coefficients of all terms, except the linear ones must be zero. As in the case above, the plan is to establish a relation between the coefficients of two different adjugates having the same set of indices. But when k is odd, the relationship is not as nice as in the even case and as such there is no general formula. However, we still have a weaker analogue of claim 3.51 for the case of k -flips.

Claim 3.52 For $s \geq 1$, if $J, \tilde{J} \in \mathcal{T}^s$, and $I^1, \dots, I^s, \tilde{I}^1, \dots, \tilde{I}^s \in \mathcal{T}^{k-1}$, where $J = \{j_1 \dots j_s\}$, $\tilde{J} = \{\tilde{j}_1 \dots \tilde{j}_s\}$, $I = \{I^1 \dots I^s\} = [I^1, \dots, I^s]$ and $\tilde{I} = \{\tilde{I}^1 \dots \tilde{I}^s\} = [\tilde{I}^1, \dots, \tilde{I}^s]$ be such that $J \cap I = \emptyset$ and (J, I) and (\tilde{J}, \tilde{I}) are related by a k -flip interchanging an index j_l in the subscript

with one from the multiindex I^m in the superscript. Also, we assume that after the interchange, the position of the multiindex containing j_l in the superscript is p and the new position of the index from the multiindex I^m in the subscript is q , i.e., $j_l \in \tilde{I}^p$ and $\tilde{j}_q \in I^m$.

Then we have,

$$\text{sgn}(J; I) (d_s)_J^I = (-1)^{(m-p)} \text{sgn}(\tilde{J}; \tilde{I}) (d_s)_{\tilde{J}}^{\tilde{I}}.$$

Since the proof of claim 3.52 is very similar to that of claim 3.51, we shall indicate only a brief sketch of the proof. Since k is odd, we deduce,

$$\begin{aligned} & \text{sgn}([j_l \bar{J}_{s-1}]; [I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]) \\ &= (-1)^{\{(a_1-1)\}} \text{sgn}(j_l, I^m) \text{sgn}(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\}) \\ & \quad \text{sgn}([j_l, I^m], [(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\})]), \\ & \text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]) \\ &= (-1)^{\{(a_2-1)\}} \text{sgn}(\tilde{j}_q, \tilde{I}^p) \text{sgn}(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\}) \\ & \quad \text{sgn}([\tilde{j}_q, \tilde{I}^p], [(\bar{J}_{s-1}; \{\bar{I}^1 \dots \bar{I}^{s-1}\})]), \end{aligned}$$

and hence, in a manner analogous to the proof of (3.49), we have,

$$\begin{aligned} & \frac{(\text{adj}_s X)_{[j_l \bar{J}_{s-1}]}^{[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]}}{\text{sgn}([j_l \bar{J}_{s-1}]; [I^m, \bar{I}^1, \dots, \bar{I}^{s-1}])} \\ &= -(-1)^{(b_1-b_2)} \frac{(\text{adj}_s X)_{[\tilde{j}_q \bar{J}_{s-1}]}^{[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]}}{\text{sgn}([\tilde{j}_q \bar{J}_{s-1}]; [\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}])}, \quad (3.51) \end{aligned}$$

for any $1 \leq s \leq s_0 + 1$, any subset $\bar{J}_{s-1} = \{\bar{j}_1, \dots, \bar{j}_{s-1}\} \subset \check{J}$ of $s-1$ indices and any choice of s multiindices $\bar{I}^1, \dots, \bar{I}^{s-1}$ out of $s_0 + 1$ multiindices, where a_1 is the position of j_l in $[j_l \bar{J}_{s-1}]$, a_2 is the position of \tilde{j}_q in $[\tilde{j}_q \bar{J}_{s-1}]$, b_1 is the position of I^m in $[I^m, \bar{I}^1, \dots, \bar{I}^{s-1}]$ and b_2 is the position of \tilde{I}^p in $[\tilde{I}^p, \bar{I}^1, \dots, \bar{I}^{s-1}]$. Claim 3.52 follows from above.

Note that claim 3.52 and claim 3.50 together now rule out the possibility that an adjugate with non-zero coefficient can have common indices between the blocks of multiindices in the superscript and proves $d_s = 0$ for all $s > \lfloor \frac{n}{k} \rfloor$. Furthermore, by claim 3.52, the coefficients of any two adjugates $(d_s)_J^I, (d_s)_{\tilde{J}}^{\tilde{I}}$ such that $I \cup J = \tilde{I} \cup \tilde{J}$, can differ only by a sign. So clearly, all of them must be 0 if one of them is. So without loss of generality, we shall restrict our attention to the coefficient of a particularly ordered adjugates, one with all distinct indices in subscript and superscripts, for which $j_1 < \dots < j_s < i_1^1 < \dots < i_{k-1}^1 < \dots < i_1^s < \dots < i_{k-1}^s$, henceforth referred to as the *totally ordered adjugate*. Hence for a given s , $2 \leq s \leq \lfloor \frac{n}{k} \rfloor$, and given $\mathcal{I} \in \mathcal{T}^{ks}$, we shall show that,

$$(d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} = 0, \quad (3.52)$$

where $j_1 < \dots < j_s < i_1^1 < \dots < i_{k-1}^1 < \dots < i_1^s < \dots < i_{k-1}^s$. To prove (3.52), we first need the following:

Claim 3.53 For any $1 \leq r \leq k-1$, we have,

$$\begin{aligned} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_r^1 i_{r+1}^2 i_{r+2}^2 \dots i_{k-1}^2\} \{i_{r+1}^1 i_{r+2}^1 \dots i_{k-1}^1 i_1^2 i_2^2 \dots i_r^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = - (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}}. \end{aligned} \quad (3.53)$$

We prove the claim by induction over r . The case for $r = 1$ follows from repeated applications of claim 3.52 as follows.

Using claim 3.52 to the k -flip interchanging j_1 and i_1^1 , then to the k -flip interchanging i_1^1 and i_1^2 and finally to the k -flip interchanging j_1 and i_1^2 , we get,

$$\begin{aligned} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = (-1)^s (d_s)_{j_2 \dots j_s i_1^1}^{\{j_1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = -(-1)^s (d_s)_{j_2 \dots j_s i_1^2}^{\{j_1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = -(-1)^s (-1)^{s-2} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_2^1 i_2^2 \dots i_{k-1}^1 i_1^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}}. \end{aligned}$$

This proves the case for $r = 1$.

We now assume that (3.53) is true for $1 \leq r \leq r_0 - 1$ and show the result for $r = r_0$. To show this, it is enough to prove that for any $2 \leq r_0 \leq k-1$,

$$\begin{aligned} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{r_0-1}^1 i_{r_0}^2 i_{r_0+1}^2 \dots i_{k-1}^2\} \{i_{r_0}^1 i_{r_0+1}^1 \dots i_{k-1}^1 i_1^2 i_2^2 \dots i_{r_0-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{r_0-1}^1 i_{r_0}^2 i_{r_0+1}^2 \dots i_{k-1}^2\} \{i_{r_0+1}^1 i_{r_0+2}^1 \dots i_{k-1}^1 i_1^2 i_2^2 \dots i_{r_0}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}}. \end{aligned} \quad (3.54)$$

Indeed the result for $r = r_0$ follows by combining the induction hypothesis and (3.54). The proof is similar to the case for $r = 1$. Indeed, by applying claim 3.52 to the k -flip interchanging j_1 and $i_{r_0}^1$, then to the k -flip interchanging $i_{r_0}^1$ and $i_{r_0}^2$ and finally to the k -flip interchanging j_1 and $i_{r_0}^2$, we deduce ,

$$\begin{aligned} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 \dots i_{r_0-1}^1 i_{r_0}^2 \dots i_{k-1}^2\} \{i_{r_0}^1 i_{r_0+1}^1 \dots i_{k-1}^1 i_1^2 \dots i_{r_0-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = (-1)^{s-1} (d_s)_{j_2 \dots j_s i_{r_0}^1}^{\{j_1 i_{r_0+1}^1 \dots i_{k-1}^1 i_1^2 \dots i_{r_0-1}^2\} \{i_1^1 \dots i_{r_0-1}^1 i_{r_0}^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = -(-1)^{s-1} (d_s)_{j_2 \dots j_s i_{r_0}^2}^{\{j_1 i_{r_0+1}^1 \dots i_{k-1}^1 i_1^2 \dots i_{r_0-1}^2\} \{i_1^1 \dots i_{r_0}^1 i_{r_0+1}^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = -(-1)^{s-1} (-1)^{s-2} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 \dots i_{r_0}^1 i_{r_0+1}^2 \dots i_{k-1}^2\} \{i_{r_0+1}^1 \dots i_{k-1}^1 i_1^2 \dots i_{r_0}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}}. \end{aligned}$$

This proves (3.54) and establishes claim 3.53.

Now, using claim 3.53, in particular for $r = k-1$, we obtain,

$$\begin{aligned} (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}} \\ = - (d_s)_{j_1 j_2 \dots j_s}^{\{i_1^1 i_2^1 \dots i_{k-1}^1\} \{i_1^2 i_2^2 \dots i_{k-1}^2\} \dots \{i_1^s i_2^s \dots i_{k-1}^s\}}. \end{aligned}$$

This proves (3.52) and finishes the proof of the lemma in the case when k is odd and thereby establishes lemma 3.47 in all cases. ■

3.5.3 Equivalence theorem

Theorem 3.54 *Let $2 \leq k \leq n$,*

$$f : \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \text{and} \quad \pi^{ext,k} : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \Lambda^k(\mathbb{R}^n)$$

be the projection map. Then the following equivalences hold

$$f \text{ ext. one convex} \Leftrightarrow f \circ \pi^{ext,k} \text{ rank one convex}$$

$$f \text{ ext. quasiconvex} \Leftrightarrow f \circ \pi^{ext,k} \text{ quasiconvex.}$$

$$f \text{ ext. polyconvex} \Leftrightarrow f \circ \pi^{ext,k} \text{ polyconvex}$$

Remark 3.55 (i) *One should not misinterpret the meaning of the theorem.*

- The theorem does not say that any quasiconvex or rank one convex function $\phi : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \mathbb{R}$ is of the form $f \circ \pi^{ext,k}$ with f ext. quasiconvex or ext. one convex as the following example shows. We let $n = k = 2$, $d \in \mathbb{R}$ and

$$\phi(\Xi) = d \det \Xi$$

which is clearly polyconvex (and thus quasiconvex and rank one convex) for every $d \in \mathbb{R}$. If $d \neq 0$, there is however no function $f : \Lambda^2 \rightarrow \mathbb{R}$ (in particular no ext. one convex and thus no ext. quasiconvex and no ext. polyconvex function f) such that $\phi = f \circ \pi^{ext,k}$. Indeed if such an f exists, we must have $d = 0$, since letting

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

we have $\pi^{ext,k}(X) = \pi^{ext,k}(Y) = 0$ and thus

$$d = \phi(X) = f(\pi^{ext,k}(X)) = f(\pi^{ext,k}(Y)) = \phi(Y) = 0.$$

- It can be that a result is false for general quasiconvex functions $\phi : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \mathbb{R}$ but is valid for functions $f \circ \pi^{ext,k} : \mathbb{R}^{\binom{n}{k-1} \times n} \rightarrow \mathbb{R}$. This has been seen on several occasions (see, for example, theorems 3.37 (ii) or 3.30 (ii)).

(ii) *The following equivalence is, of course, trivially true*

$$f \text{ convex} \Leftrightarrow f \circ \pi^{ext,k} \text{ convex.}$$

(iii) *When $k = 1$, clearly all the notions are equivalent to ordinary convexity.*

Proof (i) Recall (cf. Proposition 3.43) that

$$\pi^{\text{ext},k}(\alpha \otimes \beta) = \alpha \wedge \beta.$$

The rank one convexity of $f \circ \pi^{\text{ext},k}$ follows then at once from the ext. one convexity of f . We now prove the converse. Let $\xi \in \Lambda^k$, $\alpha \in \Lambda^{k-1}$ and $\beta \in \Lambda^1$; we have to show that

$$g : t \rightarrow g(t) = f(\xi + t \alpha \wedge \beta)$$

is convex. Since the map $\pi^{\text{ext},k}$ is onto, we can find $\Xi \in \mathbb{R}^{\binom{n}{k-1} \times n}$ so that $\pi^{\text{ext},k}(\Xi) = \xi$. Therefore

$$g(t) = f(\xi + t \alpha \wedge \beta) = f\left(\pi^{\text{ext},k}(\Xi) + t \pi^{\text{ext},k}(\alpha \otimes \beta)\right) = f\left(\pi^{\text{ext},k}(\Xi + t \alpha \otimes \beta)\right)$$

and the convexity of g follows at once from the rank one convexity of $f \circ \pi^{\text{ext},k}$.

(ii) Similarly since (cf. Proposition 3.43) $\pi^{\text{ext},k}(\nabla \omega) = d\omega$, we immediately infer the quasi-convexity of $f \circ \pi^{\text{ext},k}$ from the ext. quasiconvexity of f . The reverse implication follows also in the same manner as above.

(iii) *Step 1.* Since f is ext. polyconvex (see Proposition 3.16) we can find, for every $\alpha \in \Lambda^k$, $c_s = c_s(\alpha) \in \Lambda^{ks}$, $0 \leq 2k \leq n$, such that

$$f(\beta) \geq f(\alpha) + \sum_{s=1}^{[n/k]} \langle c_s(\alpha); \beta^s - \alpha^s \rangle, \quad \text{for every } \beta \in \Lambda^k.$$

Appealing to the proposition 3.45 we get, for every $\xi \in \mathbb{R}^{\binom{n}{k-1} \times n}$,

$$\begin{aligned} f\left(\pi^{\text{ext},k}(\eta)\right) &\geq f\left(\pi^{\text{ext},k}(\xi)\right) + \sum_{s=1}^{[n/k]} \left\langle c_s\left(\pi^{\text{ext},k}(\xi)\right); \left[\pi^{\text{ext},k}(\eta)\right]^s - \left[\pi^{\text{ext},k}(\xi)\right]^s \right\rangle \\ &= f\left(\pi^{\text{ext},k}(\xi)\right) + \sum_{s=1}^{[n/k]} \langle \tilde{c}_s(\xi); \text{adj}_s \eta - \text{adj}_s \xi \rangle \end{aligned}$$

for every $\eta \in \mathbb{R}^{\binom{n}{k-1} \times n}$, which shows that $f \circ \pi^{\text{ext},k}$ is indeed polyconvex (By theorem 5.6, part 3 in [25]).

Step 2. We now prove the reverse implication. Take $N = \binom{n}{k-1}$.

Since $f \circ \pi^{\text{ext},k}$ is polyconvex, we have (see theorem 5.6, part 3 in [25]), for every $\xi \in \mathbb{R}^{N \times n}$, there exists $d_s = d_s(\xi) \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}$ for all $0 \leq s \leq \min\{N, n\}$ such that

$$f\left(\pi^{\text{ext},k}(\eta)\right) \geq f\left(\pi^{\text{ext},k}(\xi)\right) + \sum_{s=0}^{\min\{N, n\}} \langle d_s(\xi); \text{adj}_s \eta - \text{adj}_s \xi \rangle \quad (3.55)$$

for every $\eta \in \mathbb{R}^{N \times n}$.

But this means that there exists d , given by $d = (d_1, d_2, \dots, d_{\min\{N, n\}})$ such that the function $X \mapsto g(X, d)$, where $g(X, d)$ is as defined in lemma 3.47, achieves a minima at $X = \xi$.

Then lemma 3.47 implies, for every $0 \leq s \leq \min\{N, n\}$,

$$\langle d_s, \text{adj}_s \eta - \text{adj}_s \xi \rangle = \left\langle \pi_s^{\text{ext}, k}(d_s); \pi_s^{\text{ext}, k}(\text{adj}_s \eta) - \pi_s^{\text{ext}, k}(\text{adj}_s \xi) \right\rangle \quad \text{for every } \eta \in \mathbb{R}^{N \times n}.$$

Hence, we obtain from (3.55), for every $\xi \in \mathbb{R}^{N \times n}$,

$$f\left(\pi^{\text{ext}, k}(\eta)\right) \geq f\left(\pi^{\text{ext}, k}(\xi)\right) + \sum_{s=1}^{[n/k]} \left\langle \pi_s^{\text{ext}, k}(d_s)(\xi); \pi_s^{\text{ext}, k}(\text{adj}_s \eta) - \pi_s^{\text{ext}, k}(\text{adj}_s \xi) \right\rangle \quad (3.56)$$

for every $\eta \in \mathbb{R}^{N \times n}$.

Since $\pi^{\text{ext}, k}$ is onto, given any $\alpha, \beta \in \Lambda^k$, we can find $\eta, \xi \in \mathbb{R}^{N \times n}$ such that $\pi^{\text{ext}, k}(\eta) = \beta$ and $\pi^{\text{ext}, k}(\xi) = \alpha$. Now using (3.56) and the definition of $\pi_s^{\text{ext}, k}$, we have, by defining $c_s(\alpha) = \pi_s^{\text{ext}, k}(d_s)(\xi)$, for every $\alpha \in \Lambda^k$,

$$f(\beta) \geq f(\alpha) + \sum_{s=1}^{[n/k]} \langle c_s(\alpha); \beta^s - \alpha^s \rangle, \quad \text{for every } \beta \in \Lambda^k.$$

This proves f is ext. polyconvex by virtue of Proposition 3.16 and concludes the proof of the theorem. ■

3.6 Weak lower semicontinuity and existence theorems

3.6.1 Weak lower semicontinuity

In this subsection we shall prove some easy semicontinuity results which will be enough for proving the existence theorems we need. However, the semicontinuity results can be improved considerably and this will be accomplished in the next chapter in the context of several forms.

We begin by introducing the appropriate growth condition.

Definition 3.56 (Growth condition) *Let $1 \leq k \leq n$, $1 < p < \infty$ and let $f : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ is a Carathéodory function. Then, f is said to be of growth (C_p) if for some $\alpha \geq 0$ and $1 \leq r < p$, it satisfies,*

$$-\beta(x) - \alpha|\xi|^r \leq f(x, \xi) \leq \beta(x) + g(x)|\xi|^p, \quad \text{for all } \xi \in \Lambda^k \text{ for a.e } x \in \Omega,$$

where $\beta \in L^1(\Omega)$ is nonnegative and g is a nonnegative measurable function.

Remark 3.57 *The semicontinuity results need not hold if we allow $r = p$.*

Theorem 3.58 (Sufficient condition for $1 < p < \infty$) *Let $1 \leq k \leq n$, $1 < p < \infty$ and let $f : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ be a Carathéodory function with growth (C_p) such that $\xi \mapsto f(x, \xi)$ is ext. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded open set. Let*

$\{\omega_s\} \subset L^1(\Omega; \Lambda^{k-1})$ be a sequence such that,

$$d\omega_s \rightharpoonup d\omega \text{ in } L^p(\Omega; \Lambda^k)$$

for some $\omega \in L^1(\Omega; \Lambda^{k-1})$. Then

$$\liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\omega_s) \geq \int_{\Omega} f(x, d\omega).$$

Remark 3.59 In particular, $I(\omega) = \int_{\Omega} f(x, d\omega)$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega; \Lambda^{k-1})$ and $W^{d,p}(\Omega; \Lambda^{k-1})$, for every $1 < p < \infty$ and also in $W^{d,p,q}(\Omega; \Lambda^{k-1})$, for every $1 < p < \infty$ and any $1 \leq q \leq \infty$.

Proof Using theorem 2.46, for each $s \in \mathbb{N}$, we find $\alpha_s \in W^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\begin{cases} d\alpha_s = d\omega_s & \text{and} & \delta\alpha_s = 0 & \text{in } \Omega, \\ \nu \lrcorner \alpha_s = 0 & & & \text{on } \partial\Omega. \end{cases}$$

and we have the estimate,

$$\|\alpha_s\|_{W^{1,p}} \leq C \|d\omega_s\|_{L^p}$$

for some constant $C > 0$, independent of s . Since $\{d\omega_s\}$ is weakly convergent in $L^p(\Omega; \Lambda^k)$, it follows that the sequence $\{\alpha_s\}$ is bounded in $W^{1,p}(\Omega; \Lambda^{k-1})$. Therefore, up to extraction of a subsequence that we do not relabel, there exists $\alpha \in W^{1,p}(\Omega; \Lambda^{k-1})$ such that $\alpha_s \rightharpoonup \alpha$ in $W^{1,p}(\Omega; \Lambda^{k-1})$. Note that this implies $d\omega = d\alpha$ in Ω , by uniqueness of the weak limit in L^p .

According to Theorem 3.54, we have that $X \mapsto f(x, \pi^{\text{ext},k}(X))$ is quasiconvex for every $X \in \mathbb{R}^{\binom{n}{k-1} \times n}$ for a.e $x \in \Omega$. Since f has growth (C_p) , $f(x, \pi^{\text{ext},k}(X))$ also satisfies the usual growth conditions. Then classical results (see, for example, Theorem 8.4 in [25]) show that

$$\begin{aligned} \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\omega_s) &= \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\alpha_s) = \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, \pi^{\text{ext},k}(\nabla \alpha_s)) \\ &\geq \int_{\Omega} f(x, \pi^{\text{ext},k}(\nabla \alpha)) = \int_{\Omega} f(x, d\alpha) = \int_{\Omega} f(x, d\omega) \end{aligned}$$

This completes the proof. ■

Analogously, we can show the dual results.

Theorem 3.60 (Sufficient condition for $1 < p < \infty$) Let $0 \leq k \leq n-1$, $1 < p < \infty$ and let $f : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ be a Carathéodory function with growth (C_p) such that $\xi \mapsto f(x, \xi)$ is ext. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded open set. Let $\{\omega_s\} \subset L^1(\Omega; \Lambda^{k+1})$ be a sequence such that,

$$\delta\omega_s \rightharpoonup \delta\omega \text{ in } L^p(\Omega; \Lambda^k)$$

for some $\omega \in L^1(\Omega; \Lambda^{k+1})$. Then

$$\liminf_{s \rightarrow \infty} \int_{\Omega} f(x, \delta\omega_s) \geq \int_{\Omega} f(x, \delta\omega).$$

Remark 3.61 In particular, $I(\omega) = \int_{\Omega} f(x, \delta\omega)$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega; \Lambda^{k+1})$ and $W^{\delta,p}(\Omega; \Lambda^{k+1})$, for every $1 < p < \infty$ and also in $W^{\delta,p,q}(\Omega; \Lambda^{k+1})$, for every $1 < p < \infty$ and any $1 \leq q \leq \infty$.

However, the semicontinuity result is no longer true, in general, if we have explicit dependence on ω . When $k = 1$, the spaces $W^{1,p}$ and $W^{d,p}$ coincide and the semicontinuity result holds (cf. theorem 3.23 in [25]). However, as soon as $k \geq 2$, we have the following result.

Theorem 3.62 (Counterexample to semicontinuity) Let $n \geq 2$, $2 \leq k \leq n$, $1 < p < \infty$ and let $\Omega = [0, 2\pi]^n \subset \mathbb{R}^n$. Let

$$I(\omega) := \frac{1}{p} \int_{\Omega} |d\omega|^p - \frac{1}{p} \int_{\Omega} |\omega|^p, \text{ for all } \omega \in W^{d,p}(\Omega; \Lambda^{k-1}).$$

Then I is not weakly lower semicontinuous in $W^{d,p}(\Omega; \Lambda^{k-1})$.

Proof Consider a sequence of exact forms $\{d\theta_{\nu}\} \subset L^p(\Omega; \Lambda^{k-1})$ such that

$$d\theta_{\nu} \rightharpoonup d\theta \text{ in } L^p(\Omega; \Lambda^{k-1}),$$

but

$$d\theta_{\nu} \not\rightarrow d\theta \text{ in } L^p(\Omega; \Lambda^{k-1}),$$

for some $d\theta \in L^p(\Omega; \Lambda^{k-1})$.

To construct such a sequence, it is enough to consider a sequence $\{\theta_{\nu}\} \subset W^{1,p}(\Omega; \Lambda^{k-2})$ which converges weakly to θ in $W^{1,p}(\Omega; \Lambda^{k-2})$, but not strongly. For example, define

$$\theta_{\nu} := \frac{1}{\nu} \sin(\nu x_1) e^{i_1} \wedge \dots \wedge e^{i_{k-2}},$$

where $2 \leq i_1 < \dots < i_{k-2} \leq n$, with the understanding that when $k - 2 = 0$, we just take $\theta_{\nu} := \frac{1}{\nu} \sin(\nu x_1)$. We have,

$$d\theta_{\nu} = \cos(\nu x_1) e^1 \wedge e^{i_1} \wedge \dots \wedge e^{i_{k-2}}.$$

Clearly, $\{d\theta_{\nu}\} \subset W^{d,p}(\Omega; \Lambda^{k-1})$ and $d\theta \in W^{d,p}(\Omega; \Lambda^{k-1})$ and we have,

$$d\theta_{\nu} \rightharpoonup d\theta \text{ in } L^p(\Omega; \Lambda^{k-1}), \quad \text{but } d\theta_{\nu} \not\rightarrow d\theta \text{ in } L^p(\Omega; \Lambda^{k-1}).$$

But, clearly,

$$\begin{aligned}
\liminf_{\nu \rightarrow \infty} I(d\theta_\nu) &= \liminf_{\nu \rightarrow \infty} \left(-\frac{1}{p} \int_{\Omega} |d\theta_\nu|^p \right) \\
&= -\frac{1}{p} \limsup_{\nu \rightarrow \infty} \int_{\Omega} |d\theta_\nu|^p \\
&\leq -\frac{1}{p} \liminf_{\nu \rightarrow \infty} \int_{\Omega} |d\theta_\nu|^p \\
&\leq -\frac{1}{p} \int_{\Omega} |d\theta|^p \\
&= I(d\theta).
\end{aligned}$$

But by semicontinuity,

$$\liminf_{\nu \rightarrow \infty} I(d\theta_\nu) \geq I(d\theta).$$

These two together implies,

$$\liminf_{\nu \rightarrow \infty} I(d\theta_\nu) = I(d\theta).$$

But the equality is impossible since that would imply,

$$\limsup_{\nu \rightarrow \infty} \|d\theta_\nu\|_{L^p}^p = \liminf_{\nu \rightarrow \infty} \|d\theta_\nu\|_{L^p}^p = \lim_{\nu \rightarrow \infty} \|d\theta_\nu\|_{L^p}^p = \|d\theta\|_{L^p}^p.$$

Since $d\theta_\nu \rightharpoonup d\theta$ in L^p , this implies the strong convergence in L^p , which contradicts the fact that $d\theta_\nu \not\rightarrow d\theta$ in $L^p(\Omega; \Lambda^{k-1})$. This finishes the proof. ■

Remark 3.63 (i) Note that when $k = 1$, this functional reduces to

$$I(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{p} \int_{\Omega} |u|^p, \text{ for all } u \in W^{1,p}(\Omega).$$

This is known to be weakly lower semicontinuous in $W^{1,p}$ (cf. theorem 3.23 in [25]).

(ii) Analogously, for $n \geq 2$ and $0 \leq k \leq n - 2$, the functional

$$I(\omega) = \frac{1}{p} \int_{\Omega} |\delta\omega|^p - \frac{1}{p} \int_{\Omega} |\omega|^p, \text{ for all } \omega \in W^{\delta,p}(\Omega; \Lambda^{k+1}).$$

is not weakly lower semicontinuous in $W^{\delta,p}(\Omega; \Lambda^{k+1})$.

3.6.2 Existence theorems in $W^{1,p}$

Theorem 3.64 Let $1 \leq k \leq n$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^{k-1})$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $\xi \mapsto f(x, \xi)$ is ext. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$ and verifies, for every $\xi \in \Lambda^k$,

$$c_1 |\xi|^p + \gamma_1(x) \leq f(x, \xi) \leq c_2 |\xi|^p + \gamma_2(x)$$

for a.e $x \in \Omega$ for some $c_1, c_2 > 0$ and some $\gamma_1, \gamma_2 \in L^1(\Omega)$. $g \in L^{p'}(\Omega; \Lambda^{k-1})$ be such that $\delta g = 0$ in the sense of distributions. Let

$$(\mathcal{P}_{0,ext}) \quad \inf \left\{ \int_{\Omega} [f(x, d\omega) + \langle g; \omega \rangle] : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m.$$

Then the problem $(\mathcal{P}_{0,d})$ has a minimizer.

Remark 3.65 (i) When $k = 1$, the condition $\delta g = 0$ in the sense of distributions, is automatically satisfied for all $g \in L^{p'}(\Omega)$ and hence is not a restriction.

(ii) However, as soon as $k \geq 2$, g being coclosed is a non-trivial restriction and the theorem does not hold if we drop this assumption. In fact, we can even show that if $(\mathcal{P}_{0,d})$ admits a minimizer and $2 \leq k \leq n$, then we must have $\delta g = 0$ in the sense of distributions. Indeed, suppose $\omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$ is a minimizer for $(\mathcal{P}_{0,d})$. Now if $\delta g \neq 0$, there exists a $\theta \in C_c^\infty(\Omega; \Lambda^{k-2})$ such that

$$\int_{\Omega} \langle g; d\theta \rangle \neq 0.$$

Replacin θ by $-\frac{1}{(\int_{\Omega} \langle g; d\theta \rangle)}\theta$, we can also assume that

$$\int_{\Omega} \langle g; d\theta \rangle = -1.$$

But $\omega + d\theta \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$ and we have,

$$\int_{\Omega} [f(x, d(\omega + d\theta)) + \langle g; \omega + d\theta \rangle] = \int_{\Omega} [f(x, d\omega) + \langle g; \omega \rangle] + \int_{\Omega} \langle g; d\theta \rangle = m - 1 < m,$$

which is impossible since ω is a minimizer. This establishes the necessity of the condition $\delta g = 0$.

(iii) When $k \geq 2$, let ν be the outward unit normal to $\partial\Omega$ and let

$$(\mathcal{P}_{\delta,T}) \quad \inf \left\{ \int_{\Omega} [f(x, d\omega) + \langle g; \omega \rangle] : \omega \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m_{\delta,T}$$

where $\omega \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$ stands for the set of $\omega \in W^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\delta\omega = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega.$$

The proof of the theorem will show that $(\mathcal{P}_{\delta,T})$ also have a minimizer under the hypotheses of the theorem 3.64 and that $m_{\delta,T} = m$.

(iv) Note that if $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the hypotheses of the theorem for some $1 < p < \infty$, then for any $G \in L^{p'}(\Omega; \Lambda^k)$, the function $F : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$, defined by,

$$F(x, \xi) = f(x, \xi) + \langle G(x); \xi \rangle \quad \text{for every } \xi \in \Lambda^k,$$

also satisfies all the hypotheses with the same p .

- (v) When the function f is not ext. quasiconvex, in general the problem will not have a solution. However in many cases it does have one, but the argument is of a different nature and uses results on differential inclusions, see Bandyopadhyay-Barroso-Dacorogna-Matias [9], and Dacorogna-Fonseca [26].

Proof Step 1 First we claim that we can assume $g = 0$. Since $g \in L^{p'}(\Omega; \Lambda^{k-1})$ satisfies $\delta g = 0$ in the sense of distributions, by theorem 2.43, we can find $G \in W^{1,p'}(\Omega; \Lambda^k)$, such that,

$$\begin{cases} dG = 0 & \text{and} & \delta G = g & \text{in } \Omega, \\ \nu \wedge G = 0 & & & \text{on } \partial\Omega. \end{cases}$$

Then we have, substituting and integrating by parts,

$$\int_{\Omega} \langle g; \omega \rangle = \int_{\Omega} \langle \delta G; \omega \rangle = - \int_{\Omega} \langle G; d\omega \rangle + \int_{\partial\Omega} \langle \nu \lrcorner G; \omega \rangle = - \int_{\Omega} \langle G; d\omega \rangle + \int_{\partial\Omega} \langle \nu \lrcorner G; \omega_0 \rangle.$$

Given $\omega_0 \in W^{1,p}(\Omega; \Lambda^{k-1})$ and $g \in L^{p'}(\Omega; \Lambda^{k-1})$, $\int_{\partial\Omega} \langle \nu \lrcorner G; \omega_0 \rangle$ is just a real number and thus,

$$\begin{aligned} & \inf \left\{ \int_{\Omega} [f(x, d\omega) + \langle g; \omega \rangle] : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} \\ &= \inf \left\{ \int_{\Omega} [f(x, d\omega) - \langle G; d\omega \rangle] : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} + \int_{\partial\Omega} \langle \nu \lrcorner G; \omega_0 \rangle. \end{aligned}$$

Hence finding a minimizer of $(\mathcal{P}_{0,ext})$ is equivalent to finding a minimizer of the following:

$$\inf \left\{ \int_{\Omega} F(x, d\omega) : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m',$$

where $F : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ is given by,

$$F(x, \xi) = f(x, \xi) + \langle G(x); \xi \rangle \quad \text{for every } \xi \in \Lambda^k.$$

It is easy to verify that F satisfies all the hypotheses that f satisfies. This shows the claim.

Step 2 By step 1, we assume from now on that $g = 0$. Now note that if

$$\alpha_s \rightharpoonup \alpha \quad \text{in } W^{1,p}(\Omega; \Lambda^{k-1})$$

then by theorem 3.58, we have,

$$\liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\alpha_s) \geq \int_{\Omega} f(x, d\alpha).$$

Step 3 Let ω_s be a minimizing sequence of $(\mathcal{P}_{0,d})$, i.e.

$$\int_{\Omega} f(x, d\omega_s) \rightarrow m.$$

In view of the coercivity condition, we find that there exists a constant $c_3 > 0$ such that

$$\|d\omega_s\|_{L^p} \leq c_3.$$

(i) According to Theorem 2.43, we can find $\alpha_s \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\begin{cases} d\alpha_s = d\omega_s & \text{in } \Omega \\ \delta\alpha_s = 0 & \text{in } \Omega \\ \nu \wedge \alpha_s = \nu \wedge \omega_s = \nu \wedge \omega_0 & \text{on } \partial\Omega \end{cases}$$

and there exist constants $c_4, c_5 > 0$ such that

$$\|\alpha_s\|_{W^{1,p}} \leq c_4 [\|d\omega_s\|_{L^p} + \|\omega_0\|_{W^{1,p}}] \leq c_5.$$

(ii) Therefore, up to the extraction of a subsequence that we do not relabel, there exists $\alpha \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$

$$\alpha_s \rightharpoonup \alpha \quad \text{in } W^{1,p}(\Omega; \Lambda^{k-1}).$$

(iii) We then use Theorem 2.47, to find $\omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$\begin{cases} d\omega = d\alpha & \text{in } \Omega \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

Step 4 We combine the two steps to get

$$m = \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\omega_s) = \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\alpha_s) \geq \int_{\Omega} f(x, d\alpha) = \int_{\Omega} f(x, d\omega) \geq m.$$

This concludes the proof of the theorem. ■

Remark 3.66 Unless $k = 1$, uniqueness of minimizer can not be expected even with additional assumptions like topological restrictions on the domain and strict convexity of the map $\xi \mapsto f(x, \xi)$. Firstly, if $\omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$ is a minimizer of $(\mathcal{P}_{0,ext})$, then $\omega + h$ is also a minimizer for every nontrivial harmonic field h which vanishes on $\partial\Omega$, i.e $h \in C^\infty(\Omega; \Lambda^{k-1})$ is a solution to

$$\begin{cases} dh = 0 & \text{in } \Omega, \\ \delta h = 0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{H})$$

However, even when Ω is contractible, i.e there are no nontrivial solutions to (H), $\omega + d\theta$ is a minimizer for every $\theta \in W_0^{2,p}(\Omega; \Lambda^{k-2})$ for any minimizer $\omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k-1})$. In fact, if Ω is contractible, by Poincaré lemma, i.e theorem 2.47, this implies that adding any $\alpha \in W^{1,p}(\Omega; \Lambda^{k-1})$ which satisfies $\nu \wedge \alpha = 0$ on $\partial\Omega$ and $d\alpha = 0$ in Ω to a minimizer yields another minimizer.

In exactly analogous manner, we have,

Theorem 3.67 Let $0 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^{k+1})$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $\xi \mapsto f(x, \xi)$ is int. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$ and verifies, for every $\xi \in \Lambda^k$,

$$c_1 |\xi|^p + \gamma_1(x) \leq f(x, \xi) \leq c_2 |\xi|^p + \gamma_2(x)$$

for a.e $x \in \Omega$ for some $c_1, c_2 > 0$ and some $\gamma_1, \gamma_2 \in L^1(\Omega)$. $g \in L^{p'}(\Omega; \Lambda^{k+1})$ be such that $dg = 0$ in the sense of distributions. Let

$$(\mathcal{P}_{0,int}) \quad \inf \left\{ \int_{\Omega} [f(x, \delta\omega) + \langle g; \omega \rangle] : \omega \in \omega_0 + W_0^{1,p}(\Omega; \Lambda^{k+1}) \right\} = m.$$

Then the problem $(\mathcal{P}_{0,int})$ has a minimizer.

Remark 3.68 (i) Analogously, the condition $dg = 0$ is not a restriction when $k = n-1$ and a non-trivial restriction and indeed, a necessary condition for the existence of minimizers as soon as $k \leq n-2$.

(ii) Analogue of remark 3.65(iii) holds as well. When $k \leq n-2$, let ν be the outward unit normal to $\partial\Omega$ and let

$$(\mathcal{P}_{d,N}) \quad \inf \left\{ \int_{\Omega} [f(x, \delta\omega) + \langle g; \omega \rangle] : \omega \in \omega_0 + W_{d,N}^{1,p}(\Omega; \Lambda^{k-1}) \right\} = m_{d,N}$$

where $\omega \in \omega_0 + W_{\delta,T}^{1,p}(\Omega; \Lambda^{k-1})$ stands for the set of $\omega \in W^{1,p}(\Omega; \Lambda^{k-1})$ such that

$$d\omega = 0 \text{ in } \Omega \quad \text{and} \quad \nu \lrcorner \omega = \nu \lrcorner \omega_0 \text{ on } \partial\Omega.$$

Then $(\mathcal{P}_{d,N})$ also have a minimizer under the hypotheses of the theorem 3.67 and that $m_{d,N} = m$.

(iii) Analogously, if $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the hypotheses of the theorem 3.67 for some $1 < p < \infty$, then for any $G \in L^{p'}(\Omega; \Lambda^k)$, the function $F : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$, defined by,

$$F(x, \xi) = f(x, \xi) + \langle G(x); \xi \rangle \quad \text{for every } \xi \in \Lambda^k,$$

also satisfies all the hypotheses with the same p .

(iv) Once again, uniqueness can not be expected unless $k = n-1$ even with additional assumptions like topological restrictions on the domain and strict convexity of the map $\xi \mapsto f(x, \xi)$.

Note that integrands with more general explicit dependence on ω , i.e $f(x, \omega, d\omega)$ or $f(x, \omega, \delta\omega)$ can not be handled by the above method, as the weak limit of the minimizing sequence $\{\omega_s\}$ is not the minimizer. In fact, the minimizing sequence $\{\omega_s\}$ need not have a limit point in $W^{1,p}(\Omega; \Lambda^{k-1})$ (respectively, $W^{1,p}(\Omega; \Lambda^{k+1})$), at all. On the other hand, though the minimizing sequence $\{\omega_s\}$ must have a limit point in $W^{d,p}(\Omega; \Lambda^{k-1})$ (respectively, $W^{\delta,p}(\Omega; \Lambda^{k+1})$), the functional need not be semicontinuous in $W^{d,p}(\Omega; \Lambda^{k-1})$ (respectively, $W^{\delta,p}(\Omega; \Lambda^{k+1})$), as shown by the counterexample in theorem 3.62 (respectively, remark 3.63(ii)). However, if the explicit

dependence on ω is in the form of an additive term which is convex and coercive, then existence of minimizers can be still be ensured, although in a larger space. This is the goal of the next subsection.

3.6.3 Existence theorems in $W^{d,p,q}$

Theorem 3.69 *Let $1 \leq k \leq n$, $1 < p, q < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set, $\omega_0 \in W^{d,p,q}(\Omega; \Lambda^{k-1})$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $\xi \mapsto f(x, \xi)$ is ext. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$ and verifies, for every $\xi \in \Lambda^k$,*

$$c_1 |\xi|^p + \gamma_1(x) \leq f(x, \xi) \leq c_2 |\xi|^p + \gamma_2(x)$$

for a.e $x \in \Omega$ for some $c_1, c_2 > 0$ and some $\gamma_1, \gamma_2 \in L^1(\Omega)$. Let $g : \Omega \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $u_0 \mapsto g(x, u_0)$ is convex for every $u_0 \in \Lambda^{k-1}$ for a.e $x \in \Omega$ and verifies, for every $u_0 \in \Lambda^{k-1}$,

$$g(x, u_0) \geq c_3 |u_0|^q + \gamma_3(x)$$

for a.e $x \in \Omega$ for some $c_3 > 0$ and some $\gamma_3 \in L^1(\Omega)$. Let

$$(\mathcal{P}_0) \quad \inf \left\{ \int_{\Omega} [f(x, d\omega) + g(x, \omega)] : \omega \in \omega_0 + W_T^{d,p,q}(\Omega; \Lambda^{k-1}) \right\} = m.$$

If

$$I(\omega_0) = \int_{\Omega} [f(x, d\omega_0) + g(x, \omega_0)] < \infty,$$

then the problem (\mathcal{P}_0) has a minimizer.

Proof *Step 1* Let $\{\omega_s\}$ be a minimizing sequence of (\mathcal{P}_0) , i.e.

$$\int_{\Omega} [f(x, d\omega_s) + g(x, \omega_s)] \rightarrow m.$$

In view of the coercivity condition, we find that there exist constants C_1, C_2 such that,

$$\|d\omega_s\|_{L^p} \leq C_1 \quad \text{and} \quad \|\omega_s\|_{L^q} \leq C_2.$$

But this implies, by passing to a subsequence if necessary, which we do not relabel,

$$d\omega_s \rightharpoonup \alpha \text{ in } L^p, \quad \text{and} \quad \omega_s \rightharpoonup \omega \text{ in } L^q,$$

for some $\alpha \in L^p(\Omega; \Lambda^{k-1})$ and some $\omega \in L^q(\Omega; \Lambda^{k-1})$.

Step 2 Now we will show that $\alpha = d\omega$. Since $\omega_s \in \omega_0 + W_T^{d,p,q}(\Omega; \Lambda^{k-1})$ for every s , for any $\phi \in C^\infty(\overline{\Omega}; \Lambda^{k-1})$, we have,

$$\int_{\Omega} \langle d\omega_s - d\omega_0, \phi \rangle = - \int_{\Omega} \langle \omega_s - \omega_0, \delta\phi \rangle,$$

for every s . By weak convergence of $\{d\omega_s\}$ and $\{\omega_s\}$, as $s \rightarrow \infty$, both sides of the above equation

converges to yield,

$$\int_{\Omega} \langle \alpha - d\omega_0, \phi \rangle = - \int_{\Omega} \langle \omega - \omega_0, \delta\phi \rangle.$$

Since ϕ is arbitrary, it follows that $(\omega - \omega_0) \in W_T^{d,p,q}(\Omega; \Lambda^{k-1})$ and $\alpha - d\omega_0 = d(\omega - \omega_0)$. Thus, $\omega \in \omega_0 + W_T^{d,p,q}(\Omega; \Lambda^{k-1})$ and $d\omega = \alpha$. Hence we can write,

$$d\omega_s \rightharpoonup d\omega \text{ in } L^p, \quad \text{and} \quad \omega_s \rightharpoonup \omega \text{ in } L^q.$$

Step 3 The hypothesis on f implies, by theorem 3.58, that

$$\liminf_{s \rightarrow \infty} \int_{\Omega} f(x, d\omega_s) \geq \int_{\Omega} f(x, d\omega).$$

Also, by convexity of g , (cf, theorem 1.2 in [24]) we have,

$$\liminf_{s \rightarrow \infty} \int_{\Omega} g(x, \omega_s) \geq \int_{\Omega} g(x, \omega).$$

Thus,

$$m = \liminf_{s \rightarrow \infty} \int_{\Omega} [f(x, d\omega_s) + g(x, \omega_s)] \geq \int_{\Omega} [f(x, d\omega) + g(x, \omega)] \geq m.$$

This completes the proof. ■

Similarly, we have the following for the dual situation.

Theorem 3.70 *Let $0 \leq k \leq n-1$, $1 < p, q < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set, $\omega_0 \in W^{\delta,p,q}(\Omega; \Lambda^{k+1})$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $\xi \mapsto f(x, \xi)$ is int. quasiconvex for every $\xi \in \Lambda^k$ for a.e $x \in \Omega$ and verifies, for every $\xi \in \Lambda^k$,*

$$c_1 |\xi|^p + \gamma_1(x) \leq f(x, \xi) \leq c_2 |\xi|^p + \gamma_2(x)$$

for a.e $x \in \Omega$ for some $c_1, c_2 > 0$ and some $\gamma_1, \gamma_2 \in L^1(\Omega)$. Let $g : \Omega \times \Lambda^{k+1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function such that $u_0 \mapsto g(x, u_0)$ is convex for every $u_0 \in \Lambda^{k+1}$ for a.e $x \in \Omega$ and verifies, for every $u_0 \in \Lambda^{k+1}$,

$$g(x, u_0) \geq c_3 |u_0|^q + \gamma_3(x)$$

for a.e $x \in \Omega$ for some $c_3 > 0$ and some $\gamma_3 \in L^1(\Omega)$. Let

$$(\mathcal{P}_{int,0}) \quad \inf \left\{ \int_{\Omega} [f(x, \delta\omega) + g(x, \omega)] : \omega \in \omega_0 + W_T^{\delta,p,q}(\Omega; \Lambda^{k+1}) \right\} = m.$$

If

$$I(\omega_0) = \int_{\Omega} [f(x, \delta\omega_0) + g(x, \omega_0)] < \infty,$$

then the problem $(\mathcal{P}_{int,0})$ has a minimizer.

Chapter 4

Functionals depending on exterior derivatives of several differential forms

4.1 Introduction

In this chapter, we begin our analysis of the functionals of the form

$$\int_{\Omega} f(d\omega_1, \dots, d\omega_m),$$

where $m \geq 1$ is an integer and $f : \Lambda^{k_1} \times \dots \times \Lambda^{k_m} \rightarrow \mathbb{R}$ is a continuous function, where $1 \leq k_i \leq n$ are integers for each $1 \leq i \leq m$. The functional depends on m -differential forms, $\omega_1, \dots, \omega_m$, where for each $1 \leq i \leq m$, ω_i is a $k_i - 1$ -differential form on Ω . When $m = 1$, the functional is precisely the one we studied in Chapter 3. However, for a general $m \geq 1$, if we assume $k_i = 1$ for all $1 \leq i \leq m$, this functional reduces to the functional

$$\int_{\Omega} f(\nabla u),$$

where $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function. This one is the central object of study in classical calculus of variations, $m = 1$ corresponding to the so-called ‘scalar case’ and for $m > 1$, the vectorial calculus of variations. So the functionals we study in this chapter is a generalization of both the classical calculus of variations and the calculus of variations for a single differential form. The analysis of this chapter gives us a unified viewpoint to deal with both in the same footing.

The main question, once again, centers around the appropriate notions of convexity. We introduce the appropriate notions, which are called, again for want of a better terminology, *vectorial ext. polyconvexity*, *vectorial ext. quasiconvexity* and *vectorial ext. one convexity*. However, unlike chapter 3, we do not strive towards a complete picture of implications and counter-implications regarding the relationship between these notions. Such a study can indeed be quite rewarding, as the notions are general enough to allow considerable richness, but we leave such an undertaking for the future. Our focus in this chapter would primarily be on the following two aspects,

- Study of sequential weak lower semicontinuity and weak continuity results,

- Deriving the central results of classical calculus of variations for the gradient case from our analysis.

We also see that if we allow explicit dependence on lower order terms, the case of the gradient is rather special.

4.2 Notions of Convexity

4.2.1 Definitions

We start with the different notions of convexity and affinity. However, to define all the relevant notions of convexity, we first need to introduce a notation.

Notation 4.1 Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. We denote $\Lambda^{\mathbf{k}} = \prod_{i=1}^m \Lambda^{k_i}(\mathbb{R}^n)$. Likewise, $\Lambda^{\mathbf{k}+r}$ stands for $\prod_{i=1}^m \Lambda^{k_i+r}(\mathbb{R}^n)$ for any $r \in \mathbb{Z} \setminus \{0\}$. Let $\xi = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}}$ and $|\xi| := \left(\sum_{i=1}^m |\xi_i|^2 \right)^{\frac{1}{2}}$. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \{\mathbb{N} \cup \{0\}\}^m$ be a multiindex, in the usual multiindex notations, with $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$. We denote $|\alpha| = \sum_{i=1}^m \alpha_i$ and $|\mathbf{k}\alpha| = \sum_{i=1}^m k_i \alpha_i$.

Now we define, for $|\mathbf{k}\alpha| < n$,

$$\xi^\alpha := \xi_1^{\alpha_1} \wedge \dots \wedge \xi_m^{\alpha_m},$$

where the powers on the right hand side represent wedge powers (e.g. $\xi_1^2 = \xi_1 \wedge \xi_1$). Moreover, $*\xi$ is also defined similarly, i.e. $*\xi = *\xi_1 \wedge \dots \wedge *\xi_m$ and

$$(*\xi)^\alpha := (*\xi_1)^{\alpha_1} \wedge \dots \wedge (*\xi_m)^{\alpha_m},$$

where the $*$ represents the Hodge star operator.

Notation 4.2 Also, for any integer $1 \leq s \leq n$, $T_s(\xi)$ stands for the vector with components ξ^α , where α varies over all possible choices such that $|\alpha| = s$.

Notation 4.3 Let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$. We define the spaces $L^{\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$ and $W^{1,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$, $W^{d,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$ to be the corresponding product spaces. E.g.

$$W^{d,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}}) = \prod_{i=1}^m W^{d,p_i}(\Omega, \Lambda^{k_i}).$$

They are obviously also endowed with the corresponding product norms. When $p_i = \infty$ for all $1 \leq i \leq m$, we denote the corresponding spaces by L^∞ , $W^{1,\infty}$ etc.

Notation 4.4 In the same manner, $\omega^\nu \rightharpoonup \omega$ in $W^{d,\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1})$ will stand for a shorthand of

$$\omega_i^\nu \rightharpoonup \omega_i \text{ in } W^{d,p_i}(\Omega; \Lambda^{k_i-1}) \quad (\overset{*}{\rightharpoonup} \text{ if } p_i = \infty),$$

for all $1 \leq i \leq m$, and $f(\mathbf{d}\omega^\nu) \rightharpoonup f(\mathbf{d}\omega)$ in $\mathcal{D}'(\Omega)$ will mean

$$f(d\omega_1^\nu, \dots, d\omega_m^\nu) \rightharpoonup f(d\omega_1, \dots, d\omega_m) \text{ in } \mathcal{D}'(\Omega).$$

Definition 4.5 Let $1 \leq k_i \leq n$ for all $1 \leq i \leq m$ and $f : \prod_{i=1}^m \Lambda^{k_i}(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) We say that f is vectorially ext. one convex, if the function

$$g : t \rightarrow g(t) = f(\xi_1 + t\alpha \wedge \beta_1, \xi_2 + t\alpha \wedge \beta_2, \dots, \xi_m + t\alpha \wedge \beta_m)$$

is convex for every collection of $\xi_i \in \Lambda^{k_i}$, $1 \leq i \leq m$, $\alpha \in \Lambda^1$ and $\beta_i \in \Lambda^{k_i-1}$ for all $1 \leq i \leq m$. If the function g is affine we say that f is vectorially ext. one affine.

(ii) A Borel measurable and locally bounded function f is said to be vectorially ext. quasi-convex, if

$$\int_{\Omega} f(\xi_1 + d\omega_1(x), \xi_2 + d\omega_2(x), \dots, \xi_m + d\omega_m(x)) \geq f(\xi_1, \xi_2, \dots, \xi_m) \text{ meas}(\Omega)$$

for every bounded open set Ω , for every collection of $\xi_i \in \Lambda^{k_i}$ and $\omega_i \in W_0^{1,\infty}(\Omega; \Lambda^{k_i-1})$, $1 \leq i \leq m$. If equality holds, we say that f is vectorially ext. quasilinear.

(iii) We say that f is vectorially ext. polyconvex, if there exists a convex function such that

$$f(\xi) = F(T_1(\xi), \dots, T_N(\xi)),$$

where

$$N = \left\lceil \frac{n}{\min_{1 \leq i \leq m} k_i} \right\rceil.$$

If F is affine, we say that f is vectorially ext. polyaffine.

Remark 4.6 The definition of vectorial ext. quasiconvexity already appeared in Iwaniec-Lutoborski [39], which the authors simply called quasiconvexity. In the same article, the authors also introduce another convexity notion, which they called polyconvexity. But the definition of polyconvexity introduced in Iwaniec-Lutoborski [39] is not the same as vectorial ext. polyconvexity. See remark 4.10 for more on this.

Definition 4.7 Let $0 \leq k_i \leq n-1$ for all $1 \leq i \leq m$ and $f : \prod_{i=1}^m \Lambda^{k_i}(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) We say that f is vectorially int. one convex, if the function

$$g : t \rightarrow g(t) = f(\xi_1 + t\alpha \lrcorner \beta_1, \xi_2 + t\alpha \lrcorner \beta_2, \dots, \xi_m + t\alpha \lrcorner \beta_m)$$

is convex for every collection of $\xi_i \in \Lambda^{k_i}$, $1 \leq i \leq m$, $\alpha \in \Lambda^1$ and $\beta_i \in \Lambda^{k_i+1}$ for all $1 \leq i \leq m$. If the function g is affine we say that f is vectorially int. one affine.

(ii) A Borel measurable and locally bounded function f is said to be vectorially int. quasi-

convex, if

$$\int_{\Omega} f(\xi_1 + \delta\omega_1(x), \xi_2 + \delta\omega_2(x), \dots, \xi_m + \delta\omega_m(x)) \geq f(\xi_1, \xi_2, \dots, \xi_m) \text{meas}(\Omega)$$

for every bounded open set Ω , for every collection of $\xi_i \in \Lambda^{k_i}$ and $\omega_i \in W_0^{1,\infty}(\Omega; \Lambda^{k_i+1})$, $1 \leq i \leq m$. If equality holds, we say that f is vectorially int. quasilinear.

(iii) We say that f is vectorially int. polyconvex, if there exists a convex function such that

$$f(\xi) = F(T_1(*\xi), \dots, T_N(*\xi)),$$

where

$$N = \left\lceil \frac{n}{\min_{1 \leq i \leq m} \{n - k_i\}} \right\rceil.$$

If F is affine, we say that f is vectorially int. polyaffine.

4.2.2 Main Properties

The different notions of vectorial ext. convexity are related as follows.

Theorem 4.8 Let $\mathbf{k} = (k_1, \dots, k_m)$ with $1 \leq k_i \leq n$ for all $1 \leq i \leq m$ and $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$. Then

$$\begin{aligned} f \text{ convex} &\Rightarrow f \text{ vectorially ext. polyconvex} \Rightarrow f \text{ vectorially ext. quasiconvex} \\ &\Rightarrow f \text{ vectorially ext. one convex.} \end{aligned}$$

Moreover if $f : \Lambda^{\mathbf{k}}(\mathbb{R}^n) \rightarrow \mathbb{R}$ is vectorially ext. one convex, then f is locally Lipschitz. If, in addition f is C^2 , then for every $\xi \in \Lambda^{\mathbf{k}}$, $\alpha \in \Lambda^1$ and $\beta_i \in \Lambda^{k_i-1}$ for $1 \leq i \leq m$,

$$\sum_{i,j=1}^m \sum_{\substack{I \in \mathcal{T}^{k_i} \\ J \in \mathcal{T}^{k_j}}} \frac{\partial^2 f(\xi)}{\partial \xi_{i,I} \partial \xi_{j,J}} (\alpha \wedge \beta_i)_I (\alpha \wedge \beta_j)_J \geq 0,$$

where $\xi_i = \sum_{I \in \mathcal{T}^{k_i}} \xi_{i,I} e^I$ for all $1 \leq i \leq m$.

Proof The proof is very similar to the proof of Theorem 3.10. We only mention here the essential differences. The implication that

$$f \text{ convex} \Rightarrow f \text{ vectorially ext. polyconvex}$$

is trivial.

To prove

$$f \text{ vectorially ext. polyconvex} \Rightarrow f \text{ vectorially ext. quasiconvex},$$

we once again use Jensen's inequality. The argument is exactly the same as in Theorem 3.10 as soon as we show

$$\int_{\Omega} (\xi + d\omega)^{\alpha} = \xi^{\alpha} \text{meas}(\Omega),$$

for any $\xi \in \Lambda^k$, for any $\omega \in W_0^{1,\infty}(\Omega, \Lambda^k)$ and for any multiindex $\alpha = (\alpha_1, \dots, \alpha_m) \in \{\mathbb{N} \cup \{0\}\}^m$ with $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$. We prove this using induction over $|\alpha|$. The case $|\alpha| = 1$ is trivial. So we assume $|\alpha| > 1$. Thus, there exists i such that $\alpha_i \geq 2$. Now, we have,

$$\begin{aligned} (\xi + d\omega)^{\alpha} &= \xi_i \wedge (\xi + d\omega)^{\beta} + d\omega_i \wedge (\xi + d\omega)^{\beta} \\ &= \xi_i \wedge (\xi + d\omega)^{\beta} + d \left[\omega_i \wedge (\xi + d\omega)^{\beta} \right], \end{aligned}$$

where β is a multiindex with $\beta_i = \alpha_i - 1$ and $\beta_j = \alpha_j$ for all $1 \leq j \leq m, i \neq j$. Since $|\beta| = |\alpha| - 1$, integrating the above and using induction for the first integral and the fact that $\omega_i = 0$ on $\partial\Omega$ for the second, we obtain the result.

To prove

$$f \text{ vectorially ext. quasiconvex} \Rightarrow f \text{ vectorially ext. one convex},$$

we also proceed in the same lines as in Theorem 3.10. For any $\lambda \in [0, 1]$, we find, using Lemma 3.7, for each $1 \leq i \leq m$, we find disjoint open sets $\Omega_1^i, \Omega_2^i \subset \Omega$ and a function $\phi^i \in W_0^{1,\infty}(\Omega; \Lambda^{k_i-1})$ such that

1. $|\text{meas}(\Omega_1^i) - \lambda \text{meas}(\Omega)| \leq \epsilon$ and $|\text{meas}(\Omega_2^i) - (1 - \lambda) \text{meas}(\Omega)| \leq \epsilon$,
2. $\|\phi^i\|_{L^\infty(\bar{\Omega})} \leq \infty$,
3. $d\phi^i(x) = \begin{cases} (1 - \lambda)(t - s)a \wedge b, & \text{if } x \in \Omega_1^i, \\ -\lambda(t - s)a \wedge b, & \text{if } x \in \Omega_2^i. \end{cases}$

Define

$$\Omega_1 = \bigcap_{i=1}^m \Omega_1^i \quad \text{and} \quad \Omega_2 = \bigcap_{i=1}^m \Omega_2^i.$$

Since this implies

$$\text{meas}(\Omega \setminus (\Omega_1 \cup \Omega_2)) \leq \lambda \text{meas}(\Omega) - \text{meas}(\Omega_1^i) + (1 - \lambda) \text{meas}(\Omega) - \text{meas}(\Omega_2^i) \leq 2\epsilon,$$

the proof follows.

The fact that f is locally Lipschitz follows once again from the observation that any vectorially ext. one convex function is separately convex. Now if f is C^2 , the function

$$g : t \rightarrow g(t) = f(\xi_1 + t\alpha \wedge \beta_1, \xi_2 + t\alpha \wedge \beta_2, \dots, \xi_m + t\alpha \wedge \beta_m)$$

is convex and C^2 . The claim follows from the fact that $g''(0) \geq 0$. ■

We can have another formulation of vectorial ext. polyconvexity. The proof of which is similar to Proposition 3.16 (see also Theorem 5.6 in [25]) and is omitted.

Proposition 4.9 Let $\mathbf{k} = (k_1, \dots, k_m)$ with $1 \leq k_i \leq n$ for all $1 \leq i \leq m$ and $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$. Then, the function f is ext. polyconvex if and only if, for every $\xi \in \Lambda^{\mathbf{k}}$, there exist $c_\alpha = c_\alpha(\xi) \in \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)$, for every $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |\mathbf{k}\alpha| \leq n$, such that

$$f(\eta) \geq f(\xi) + \sum_{\alpha} \langle c_\alpha(\xi); \eta^\alpha - \xi^\alpha \rangle, \quad \text{for every } \eta \in \Lambda^{\mathbf{k}}.$$

Remark 4.10 This formulation of the definition is better suited for comparison with the definition of polyconvexity introduced in definition 10.1 in Iwaniec-Lutoborski [39], one easily sees that their definition allows only the case $\alpha_i \in \{0, 1\}$ for all $1 \leq i \leq m$. We remark that unless k_i s are all odd integers, these two classes of polyconvex functions do not coincide and ours is strictly larger. For example, the function $f_1 : \Lambda^{k_1} \times \Lambda^{k_2} \rightarrow \mathbb{R}$ given by,

$$f_1(\xi_1, \xi_2) = \langle c; \xi_1 \wedge \xi_2 \rangle \quad \text{for every } \xi_1 \in \Lambda^{k_1}, \xi_2 \in \Lambda^{k_2}$$

where $c \in \Lambda^{k_1+k_2}$ is a constant, is polyaffine in the sense of Iwaniec-Lutoborski [39] and also vectorially ext. polyaffine. However, the function $f_2 : \Lambda^{k_1} \times \Lambda^{k_2} \rightarrow \mathbb{R}$ given by,

$$f_2(\xi_1, \xi_2) = \langle c; \xi_1 \wedge \xi_1 \rangle \quad \text{for every } \xi_1 \in \Lambda^{k_1}, \xi_2 \in \Lambda^{k_2}$$

where $c \in \Lambda^{2k_1}$ is a constant, is vectorially ext. polyaffine, but not polyaffine in the sense of Iwaniec-Lutoborski [39]. Note also that it is easy to see, by integrating by parts that both f_1 and f_2 are vectorially ext. quasilinear and hence are also quasilinear in the sense of Iwaniec-Lutoborski [39]. Also, when $m = 1$, i.e there is only one differential form, reducing the problem to the form (1.1), their definition of polyconvexity coincide with usual convexity. On the other hand, when $m = 1$, vectorial ext. polyconvexity reduces to ext. polyconvexity, which is much weaker than convexity.

We finish this section with another result which says that when $k_i = 1$ for all $1 \leq i \leq m$, the notions of vectorial ext. polyconvexity, vectorial ext. quasiconvexity and vectorial ext. one convexity are exactly the notions of polyconvexity, quasiconvexity and rank one convexity, respectively.

Proposition 4.11 Let $\mathbf{k} = (k_1, \dots, k_m)$ with $1 \leq k_i \leq n$ for all $1 \leq i \leq m$ and $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$. If $k_i = 1$ for all $1 \leq i \leq m$, then for each $\xi \in \Lambda^{\mathbf{k}}$, by identifying $\xi_i \in \Lambda^1$ as the i -th row, ξ can be written as a $m \times n$ matrix. With this identification, it follows that,

$$\begin{aligned} f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R} \text{ is vectorially ext. polyconvex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is polyconvex,} \\ f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R} \text{ is vectorially ext. quasiconvex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is quasiconvex,} \\ f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R} \text{ is vectorially ext. one convex} &\Leftrightarrow f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is rank one convex.} \end{aligned}$$

Proof The first conclusion is immediate as soon as we note that in this case, the adjugates of the matrix is precisely the wedge powers of the rows. The conclusion is about quasiconvexity is obvious from the definitions. For the conclusion about rank one convexity, note that for any

1-form α and 0-forms β_1, \dots, β_m , we can identify α with a vector in \mathbb{R}^n and we can define the vector in \mathbb{R}^m as

$$\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m.$$

Then, we have, for any $t \in \mathbb{R}$,

$$\Lambda^{\mathbf{k}} \ni (\xi_1 + t\alpha \wedge \beta_1, \dots, \xi_m + t\alpha \wedge \beta_m) = (\xi + t\alpha \otimes \beta) \in \mathbb{R}^{m \times n},$$

where ξ stands for $\boldsymbol{\xi}$, written as a $m \times n$ matrix. This concludes the proof. ■

4.3 Vectorially ext. quasilinear functions

Theorem 4.12 Let $\mathbf{k} = (k_1, \dots, k_m)$ with $1 \leq k_i \leq n$ for all $1 \leq i \leq m$ and $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$. The following statements are then equivalent.

(i) f is vectorially ext. polyaffine.

(ii) f is vectorially ext. quasilinear.

(iii) f is vectorially ext. one affine.

(iv) There exist $c_\alpha \in \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)$, for every $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |\mathbf{k}\alpha| \leq n$, such that for every $\boldsymbol{\xi} \in \Lambda^{\mathbf{k}}$,

$$f(\boldsymbol{\xi}) = \sum_{\substack{\alpha, \\ 0 \leq |\mathbf{k}\alpha| \leq n}} \langle c_\alpha; \boldsymbol{\xi}^\alpha \rangle.$$

Remark 4.13 If $k_i = 1$ for all $1 \leq i \leq m$, then this theorem recovers the characterization theorem for quasilinear functions in classical vectorial calculus of variation as a special case. Indeed, let $X \in \mathbb{R}^{m \times n}$ be a matrix, then setting $\xi_i = \sum_{j=1}^n X_{ij} e^j$ for all $1 \leq i \leq m$, we recover exactly the classical results (cf. Theorem 5.20 in [25]).

Proof (i) \Rightarrow (ii) \Rightarrow (iii) follows from Theorem 4.8. (iv) \Rightarrow (i) is immediate from the definition of vectorially ext. polyconvexity. So we only need to show (iii) \Rightarrow (iv).

We show this by induction on m . Clearly, for $m = 1$, this is just the characterization theorem for ext. one affine functions, given in theorem 3.20. We assume the result to be true for $m \leq p-1$ and show it for $m = p$. Now since f is vectorially ext. one affine, it is separately vectorially ext. one affine and using ext. one affinity with respect to ξ_p , keeping the other variables fixed, we obtain,

$$f(\boldsymbol{\xi}) = \sum_{s=1}^{\left\lfloor \frac{n}{k_p} \right\rfloor} \langle c_s(\xi_1, \dots, \xi_{p-1}); \xi_p^s \rangle,$$

where for each $1 \leq s \leq \left\lfloor \frac{n}{k_p} \right\rfloor$, the functions $c_s : \prod_{i=1}^{p-1} \Lambda^{k_i} \rightarrow \Lambda^{sk_p}$ are such that the map $(\xi_1, \dots, \xi_{p-1}) \mapsto f(\xi_1, \dots, \xi_{p-1}, \xi_p)$ is vectorially ext. one affine for any $\xi_p \in \Lambda^{k_p}$. Arguing

by degree of homogeneity, this implies that for each $1 \leq s \leq [\frac{n}{k_p}]$, every component c_S^I is vectorially ext. one affine, i.e. $(\xi_1, \dots, \xi_{p-1}) \mapsto c_s^I(\xi_1, \dots, \xi_{p-1})$ is vectorially ext. one affine for any $I \in \mathcal{T}_{sk_p}$. Applying the induction hypothesis to each of these components and multiplying out, we indeed obtain the desired result. ■

4.4 Weak lower semicontinuity

4.4.1 Necessary condition

We first show that vectorial ext. quasiconvexity is indeed a necessary condition for sequential weak lower semicontinuity of the functional of the form

$$\int_{\Omega} f(x, \omega, d\omega).$$

The proof of this result is very similar to the classical result for the gradient case (cf. Theorem 3.15 in [25]).

Theorem 4.14 (Necessary condition) *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $f : \Omega \times \Lambda^{k-1} \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying, for almost all $x \in \Omega$ and for all $(\omega, \xi) \in \Lambda^{k-1} \times \Lambda^{\mathbf{k}}$,*

$$|f(x, \omega, \xi)| \leq a(x) + b(\omega, \xi), \quad (4.1)$$

where $a \in L^1(\mathbb{R}^n)$, $b \in C(\Lambda^{k-1} \times \Lambda^{\mathbf{k}})$ is non-negative. Let $I : W^{d,\infty}(\Omega; \Lambda^{k-1}) \rightarrow \mathbb{R}$ defined by

$$I(\omega) := \int_{\Omega} f(x, \omega(x), d\omega(x)) \, dx, \text{ for all } \omega \in W^{d,\infty}(\Omega; \Lambda^{k-1}),$$

is weak * lower semicontinuous in $W^{d,\infty}(\Omega; \Lambda^{k-1})$. Then, for almost all $x_0 \in \Omega$ and for all $\omega_0 \in \Lambda^{k-1}$, $\xi_0 \in \Lambda^{\mathbf{k}}$ and $\phi \in W^{d,\infty}(D; \Lambda^{\mathbf{k}})$,

$$\int_D f(x_0, \omega_0, \xi_0 + d\phi(x)) \, dx \geq f(x_0, \omega_0, \xi_0),$$

where $D := (0, 1)^n \subset \mathbb{R}^n$. In particular, $\xi \mapsto f(x, \omega, \xi)$ is vectorially ext. quasiconvex for a.e. $x \in \Omega$ and for every $\omega \in \Lambda^{k-1}$.

Remark 4.15 *Since I being weak * lower semicontinuous in $W^{d,\infty}(\Omega; \Lambda^{k-1})$ is a necessary condition for I to be weak lower semicontinuous in $W^{d,p}(\Omega; \Lambda^{k-1})$ for any $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$, f being vectorially ext. quasiconvex is a necessary condition for weak lower semicontinuity in $W^{d,p}(\Omega; \Lambda^{k-1})$ as well.*

Proof Let $\omega_0 \in \Lambda^{k-1}$, $\xi_0 \in \Lambda^{\mathbf{k}}$ and $\phi \in W^{d,\infty}(D; \Lambda^{k-1})$ be given. Let us choose affine $\bar{\omega} \in C^\infty(\mathbb{R}^n; \Lambda^{k-1})$ such that

$$\bar{\omega}(x_0) = \omega_0, \quad d\bar{\omega} = \xi_0 \text{ in } \mathbb{R}^n.$$

Let us define

$$\begin{aligned}\lambda &= \lambda(\boldsymbol{\omega}_0, \boldsymbol{\xi}_0, \phi) := \|\phi\|_{W^{d,\infty}(D; \mathbf{\Lambda}^{k-1})} + |\boldsymbol{\xi}_0| + \|\overline{\boldsymbol{\omega}}\|_{L^\infty(\bar{\Omega}; \mathbf{\Lambda}^k)}, \\ B_\lambda &:= \left\{ (\boldsymbol{\omega}, \boldsymbol{\xi}) \in \mathbf{\Lambda}^{k-1} \times \mathbf{\Lambda}^k : |\boldsymbol{\omega}| + |\boldsymbol{\xi}| \leq \lambda \right\}, \\ \gamma &:= \max \{b(\boldsymbol{\omega}, \boldsymbol{\xi}) : (\boldsymbol{\omega}, \boldsymbol{\xi}) \in B_\lambda\}.\end{aligned}\tag{4.2}$$

For every $\nu \in \mathbb{N}$ and $\epsilon > 0$, we find a compact set $K_\nu \subset \Omega$ and continuous $f_\nu : \mathbb{R}^n \times \mathbf{\Lambda}^{k-1} \times \mathbf{\Lambda}^k \rightarrow \mathbb{R}$ such that $f : K_\nu \times B_\lambda \rightarrow \mathbb{R}$ is continuous and

$$\text{meas}(\Omega \setminus K_\nu) < \frac{1}{\nu} \text{ and } \int_{\Omega \setminus K_\nu} (a(x) + \gamma) dx < \epsilon.\tag{4.3}$$

Furthermore,

1. $f_\nu = f$ in $K_\nu \times B_\lambda$.
2. $\|f_\nu\|_{C(\mathbb{R}^n \times \mathbf{\Lambda}^{k-1} \times \mathbf{\Lambda}^k)} = \|f\|_{C(K_\nu \times B_\lambda)}$.
3. For all $\boldsymbol{\omega} \in W^{d,\infty}(\Omega; \mathbf{\Lambda}^{k-1})$,

$$\int_{\Omega \setminus K_\nu} |f_\nu(x, \boldsymbol{\omega}(x), d\boldsymbol{\omega}(x))| dx \leq \epsilon.\tag{4.4}$$

Let us write

$$\Omega_0 := \bigcap_{\nu \in \mathbb{N}} \left\{ x \in \bigcup_{j \in \mathbb{N}} K_j : x \text{ is a Lebesgue point of } \chi_{K_\nu} \text{ and } a\chi_{\Omega \setminus K_\nu} \right\}.$$

Note that $\text{meas}(\Omega \setminus \Omega_0) = 0$. Let $x_0 \in \Omega_0$ be fixed. For all $s \in \mathbb{N}$, let us write $Q_s := x_0 + \frac{1}{s}D$. We choose $s \in \mathbb{N}$ sufficiently large, say $s \geq s_0$, for which $Q_s \subset \Omega$. Extending ϕ by periodicity with respect to D to \mathbb{R}^n , for all $r \in \mathbb{N}$ and $s \geq s_0$, we define $\phi_{r,s} \in W^{d,\infty}(\Omega; \mathbf{\Lambda}^{k-1})$ by

$$\phi_{r,s}(x) := \begin{cases} \frac{1}{rs} \phi(rs(x - x_0)), & \text{if } x \in Q_s, \\ 0, & \text{if } x \in \Omega \setminus Q_s. \end{cases}$$

Note that, for each $s \in \mathbb{N}$ with $s \geq s_0$,

$$\phi_{r,s} \xrightarrow{*} 0 \text{ in } W^{d,\infty}(\Omega; \mathbf{\Lambda}^{k-1}), \text{ as } r \rightarrow \infty.$$

On defining, for each $r \in \mathbb{N}$ and $s \in \mathbb{N}$ with $s \geq s_0$, $\boldsymbol{\omega}_{r,s} \in W^{d,\infty}(\Omega; \mathbf{\Lambda}^k)$ as

$$\boldsymbol{\omega}_{r,s}(x) := \overline{\boldsymbol{\omega}}(x) + \phi_{r,s}(x), \text{ for all } x \in \Omega,$$

we note that, for each $r \in \mathbb{N}$ and $s \in \mathbb{N}$ with $s \geq s_0$,

$$(\overline{\boldsymbol{\omega}}(x), d\overline{\boldsymbol{\omega}}(x)), (\boldsymbol{\omega}_{r,s}(x), d\boldsymbol{\omega}_{r,s}(x)) \in B_\lambda, \text{ for a.e } x \in \Omega.\tag{4.5}$$

Moreover, for each $s \in \mathbb{N}$ with $s \geq s_0$,

$$\omega_{r,s} \rightarrow \bar{\omega} \text{ in } L^\infty(\Omega; \Lambda^{k-1}) \text{ and } \omega_{r,s} \xrightarrow{*} \bar{\omega} \text{ in } W^{d,\infty}(\Omega; \Lambda^{k-1}), \text{ as } r \rightarrow \infty. \quad (4.6)$$

Let us split Q_s into subcubes $Q_{s,j}^r$ of edge-length $\frac{1}{rs}$ and let x_j be the vertex of $Q_{s,j}^r$ closest to x_0 , for all $0 \leq j \leq r^n - 1$. Then, we have that

$$Q_{s,j}^r = x_j + \frac{1}{rs}D \text{ and } \bar{Q}_s = \bigcup_{j=1}^{r^n-1} \bar{Q}_{s,j}^r. \quad (4.7)$$

For each $\nu, r \in \mathbb{N}$ and $s \geq s_0$, let us define

$$\begin{aligned} I_1^{r,s}(\nu) &:= \sum_{j=1}^{r^n-1} \int_{Q_{s,j}^r} f_\nu(x_j, \bar{\omega}(x_j), d\omega_{r,s}(x)) dx, \\ I_2^{r,s}(\nu) &:= \sum_{j=1}^{r^n-1} \int_{Q_{s,j}^r} [f_\nu(x, \omega_{r,s}(x), d\omega_{r,s}(x)) - f_\nu(x_j, \bar{\omega}(x_j), d\omega_{r,s}(x))] dx, \\ I_3^{r,s}(\nu) &:= \int_{Q_s} [f(x, \omega_{r,s}(x), d\omega_{r,s}(x)) - f_\nu(x, \omega_{r,s}(x), d\omega_{r,s}(x))] dx. \end{aligned}$$

Note that, for all $\nu, r \in \mathbb{N}$ and $s \geq s_0$,

$$I(\omega_{r,s}) = \int_{\Omega \setminus Q_s} f(x, \bar{\omega}(x), d\bar{\omega}(x)) dx + I_1^{r,s}(\nu) + I_2^{r,s}(\nu) + I_3^{r,s}(\nu).$$

We now estimate each term.

Step 1. *Estimation of $I_1^{r,s}(\nu)$.*

Note that, for all $\nu, r \in \mathbb{N}$ and $s \geq s_0$,

$$\begin{aligned} I_1^{r,s}(m) &= \sum_{j=1}^{r^n-1} \int_{Q_{s,j}^r} f_\nu(x_j, \bar{\omega}(x_j), d\omega_{r,s}(x)) dx \\ &= \sum_{j=1}^{r^n-1} \int_{x_j + \frac{1}{rs}D} f_\nu(x_j, \bar{\omega}(x_j), \xi_0 + d\phi(rs(x - x_0))) dx \\ &= \sum_{j=1}^{r^n-1} \frac{1}{(rs)^n} \int_D f_\nu(x_j, \bar{\omega}(x_j), \xi_0 + d\phi(y + rs(x_j - x_0))) dy \\ &= \sum_{j=1}^{r^n-1} \frac{1}{(rs)^n} \int_D f_\nu(x_j, \bar{\omega}(x_j), \xi_0 + d\phi(y)) dy. \end{aligned}$$

Therefore, for all $\nu \in \mathbb{N}$ and $s \geq s_0$,

$$\lim_{r \rightarrow \infty} I_1^{r,s}(m) = \int_{Q_s} \left(\int_D f_\nu(x, \bar{\omega}(x), \xi_0 + d\phi(y)) dy \right) dx. \quad (4.8)$$

Step 2. *Estimation of $I_2^{r,s}(m)$.*

Since f_ν is uniformly continuous on $\overline{Q_s} \times B_\lambda$, using Equations (4.5) and (4.6), it follows that for all $\nu \in \mathbb{N}$ and $s \geq s_0$,

$$\lim_{r \rightarrow \infty} I_2^{r,s}(\nu) = 0. \quad (4.9)$$

Step 2. *Estimation of $I_3^{r,s}(\nu)$.*

Note that, for all $\nu, r \in \mathbb{N}$ and $s \geq s_0$, using the bound (4.1) on f and using Equations (4.5), (4.2), (4.3), (4.4),

$$\begin{aligned} I_3^{r,s}(\nu) &\leq \int_{\Omega} |f(x, \omega_{r,s}(x), d\omega_{r,s}(x)) - f_\nu(x, \omega_{r,s}(x), d\omega_{r,s}(x))| dx \\ &= \int_{\Omega \setminus K_\nu} |f(x, \omega_{r,s}(x), d\omega_{r,s}(x)) - f_\nu(x, \omega_{r,s}(x), d\omega_{r,s}(x))| dx \\ &= \int_{\Omega \setminus K_\nu} (|f(x, \omega_{r,s}(x), d\omega_{r,s}(x))| + |f_\nu(x, \omega_{r,s}(x), d\omega_{r,s}(x))|) dx \\ &\leq \int_{\Omega \setminus K_\nu} (a(x) + \gamma) dx + \epsilon < 2\epsilon. \end{aligned} \quad (4.10)$$

We now use Equations (4.8), (4.9), (4.10), (4.6), and the weak lower semicontinuity of I , to deduce that, for all $\nu \in \mathbb{N}$ and $s \geq s_0$,

$$\begin{aligned} &\int_{\Omega \setminus Q_s} f(x, \overline{\omega}(x), d\overline{\omega}(x)) dx + \int_{Q_s} \left(\int_D f_\nu(x, \overline{\omega}(x), \xi_0 + d\phi(y)) dy \right) dx + 2\epsilon \\ &\geq \int_{\Omega \setminus Q_s} f(x, \overline{\omega}(x), d\overline{\omega}(x)) dx + \lim_{r \rightarrow \infty} I_1^{r,s}(m) + \lim_{r \rightarrow \infty} I_2^{r,s}(m) + \limsup_{r \rightarrow \infty} I_3^{r,s}(m) \\ &\geq \limsup_{r \rightarrow \infty} \left(\int_{\Omega \setminus Q_s} f(x, \overline{\omega}(x), d\overline{\omega}(x)) dx + I_1^{r,s}(m) + I_2^{r,s}(m) + I_3^{r,s}(m) \right) \\ &\geq \liminf_{r \rightarrow \infty} I(\omega_{r,s}) \geq I(\overline{\omega}) = \int_{\Omega} f(x, \overline{\omega}(x), d\overline{\omega}(x)) dx. \end{aligned}$$

Letting $\nu \rightarrow \infty$, we deduce that, for all $s \geq s_0$ and $\epsilon > 0$,

$$\int_{Q_s} \left(\int_D f(x, \overline{\omega}(x), \xi_0 + d\phi(y)) dy \right) dx + 2\epsilon \geq \int_{Q_s} f(x, \overline{\omega}(x), d\overline{\omega}(x)) dx.$$

Therefore, for all $s \geq s_0$,

$$\int_{Q_s} \left(\int_D f(x, \overline{\omega}(x), \xi_0 + d\phi(y)) dy \right) dx \geq \int_{Q_s} f(x, \overline{\omega}(x), \xi_0) dx. \quad (4.11)$$

Let us define $F : \Omega \rightarrow \mathbb{R}$ by

$$F(x) := \int_D f(x, \overline{\omega}(x), \xi_0 + d\phi(y)) dy - f(x, \overline{\omega}(x), \xi_0), \text{ for all } x \in \Omega.$$

It remains to show that

$$F(x_0) \geq 0. \quad (4.12)$$

Since f is Carathéodory and satisfies (4.1), it follows from Equation (4.2) that

$$\begin{aligned}
|F(x)| &\leq \int_D |f(x, \overline{\omega}(x), \xi_0 + d\phi(y))| dy + |f(x, \overline{\omega}(x), \xi_0)| \\
&\leq \int_D (a(x) + b(\overline{\omega}(x), \xi_0 + d\phi(y))) dy + a(x) + b(\overline{\omega}(x), \xi_0) \\
&\leq 2(a(x) + \gamma), \text{ for all } x \in \Omega.
\end{aligned} \tag{4.13}$$

For each $\nu \in \mathbb{N}$, since f is continuous on $K_\nu \times B_\lambda$, it follows that

$$\lim_{s \rightarrow \infty} \frac{1}{\text{meas}(Q_s \cap K_\nu)} \int_{Q_s \cap K_\nu} F(x) dx = F(x_0). \tag{4.14}$$

Since, for each $\nu \in \mathbb{N}$ and $s \geq s_0$,

$$\begin{aligned}
\frac{1}{\text{meas}(Q_s)} \int_{Q_s \cap K_\nu} F(x) dx &= \frac{\text{meas}(Q_s \cap K_\nu)}{\text{meas}(Q_s)} \frac{1}{\text{meas}(Q_s \cap K_\nu)} \int_{Q_s \cap K_\nu} F(x) dx \\
&= \left(\frac{1}{\text{meas}(Q_s)} \int_{Q_s} \chi_{K_\nu}(x) dx \right) \left(\frac{1}{\text{meas}(Q_s \cap K_\nu)} \int_{Q_s \cap K_\nu} F(x) dx \right),
\end{aligned}$$

it follows from Equation (4.14) and the fact that $x_0 \in \Omega_0$,

$$\lim_{s \rightarrow \infty} \frac{1}{\text{meas}(Q_s)} \int_{Q_s \cap K_\nu} F(x) dx = \chi_{K_\nu}(x_0) F(x_0) = F(x_0), \text{ for all } m \in \mathbb{N}. \tag{4.15}$$

Furthermore, for all $\nu \in \mathbb{N}$ and $s \geq s_0$, using Equation (4.13) we deduce that,

$$\begin{aligned}
\left| \frac{1}{\text{meas}(Q_s)} \int_{Q_s \setminus (Q_s \cap K_\nu)} F(x) dx \right| &= \left| \frac{1}{\text{meas}(Q_s)} \int_{Q_s} F(x) \chi_{\Omega \setminus K_\nu}(x) dx \right| \\
&\leq \frac{2}{\text{meas}(Q_s)} \int_{Q_s} (a(x) + \gamma) \chi_{\Omega \setminus K_\nu}(x) dx
\end{aligned} \tag{4.16}$$

Therefore, for all $\nu \in \mathbb{N}$,

$$\lim_{s \rightarrow \infty} \left| \frac{1}{\text{meas}(Q_s)} \int_{Q_s \setminus (Q_s \cap K_\nu)} F(x) dx \right| \leq 2(a(x_0) + \gamma) \chi_{\Omega \setminus K_\nu}(x_0) = 0.$$

Hence, it follows from Equations (4.15) and (4.11) that, for all $\nu \in \mathbb{N}$,

$$\begin{aligned}
F(x_0) &= \lim_{s \rightarrow \infty} \frac{1}{\text{meas}(Q_s)} \int_{Q_s \cap K_\nu} F(x) dx + \lim_{s \rightarrow \infty} \frac{1}{\text{meas}(Q_s)} \int_{Q_s \setminus (Q_s \cap K_\nu)} F(x) dx \\
&= \lim_{s \rightarrow \infty} \frac{1}{\text{meas}(Q_s)} \int_{Q_s} F(x) dx \geq 0,
\end{aligned}$$

which proves Equation (4.12). This proves the theorem. ■

4.4.2 Sufficient condition

Lower semicontinuity for quasiconvex functions without lower order terms

We start by defining the growth conditions that we need.

Definition 4.16 (Growth condition) Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$.

f is said to be of growth $(\mathcal{C}_{\mathbf{p}})$, if for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}}$, f satisfies,

$$-\alpha \left(1 + \sum_{i=1}^m G_i^l(\xi_i) \right) \leq f(\boldsymbol{\xi}) \leq \alpha \left(1 + \sum_{i=1}^m G_i^u(\xi_i) \right), \quad (\mathcal{C}_{\mathbf{p}})$$

where $\alpha > 0$ is a constant and the functions G_i^l s in the lower bound and the functions G_i^u s in the upper bound has the following form:

- If $p_i = 1$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$G_i^l(\xi_i) = \alpha_i |\xi_i|^{q_i}$$

and

$$G_i^u(\xi_i) = \alpha_i |\xi_i|^{p_i},$$

for some $1 \leq q_i < p_i$ and for some constant $\alpha_i \geq 0$.

- If $p_i = \infty$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \eta_i(|\xi_i|).$$

for some nonnegative, continuous, increasing function η_i .

Now we derive a lemma which is essentially an analogue of the result relating quasiconvexity with $W^{1,p}$ -quasiconvexity in the classical case (see [8]).

Lemma 4.17 ($W^{d,p}$ -vectorial ext. quasiconvexity) Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ satisfy, for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}}$,

$$f(\boldsymbol{\xi}) \leq \alpha \left(1 + \sum_{i=1}^m G_i^u(\xi_i) \right),$$

where $\alpha > 0$ is a constant and the functions G_i^u s has the following form, as defined above, i.e.,

- If $p_i = 1$, then,

$$G_i^u(\xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$G_i^u(\xi_i) = \alpha_i |\xi_i|^{p_i},$$

for some constant $\alpha_i \geq 0$.

- If $p_i = \infty$, then,

$$G_i^u(\xi_i) = \eta_i(|\xi_i|).$$

for some nonnegative, continuous, increasing function η_i .

Then the following are equivalent.

(i) f is vectorially ext. quasiconvex.

(ii) For every \mathbf{q} such that $p_i \leq q_i \leq \infty$ for every $i = 1, \dots, m$, we have,

$$\frac{1}{\text{meas}(\Omega)} \int_{\Omega} f(\boldsymbol{\xi} + d\phi) \geq f(\boldsymbol{\xi}),$$

for every $\phi \in W_T^{d,\mathbf{q}}(\Omega; \mathbf{\Lambda}^{k-1})$.

Proof (ii) implies (i) is trivial. So we only need to show (i) implies (ii). So we assume $f : \mathbf{\Lambda}^k \rightarrow \mathbb{R}$ is vectorially ext. quasiconvex.

Now we claim that for any $\phi \in W_T^{d,\mathbf{q}}(\Omega; \mathbf{\Lambda}^{k-1}) \subset W_T^{d,\mathbf{p}}(\Omega; \mathbf{\Lambda}^{k-1})$, we can find $\phi^\nu \in C_c^\infty(\Omega; \mathbf{\Lambda}^{k-1})$ such that $\{\phi^\nu\}$ is uniformly bounded in $W^{d,\mathbf{p}}(\Omega; \mathbf{\Lambda}^{k-1})$ and $d\phi^\nu \rightarrow d\phi$ for a.e. $x \in \Omega$.

Indeed, if $p_i < \infty$, then we can actually find $\{\phi_i^\nu\} \subset C_c^\infty(\Omega; \mathbf{\Lambda}^{k-1})$ with $\phi_i^\nu \rightarrow \phi_i$ in $W^{d,p_i}(\Omega; \mathbf{\Lambda}^{k_i-1})$, which clearly implies what we claimed. If $p_i = \infty$, then by the usual trick of truncating and mollifying, we easily find a sequence $\{\phi_i^\nu\} \in C_c^\infty(\Omega; \mathbf{\Lambda}^{k-1})$, which is uniformly bounded in $W^{d,\infty}(\Omega; \mathbf{\Lambda}^{k_i-1})$ and $d\phi_i^\nu \rightarrow d\phi_i$ in $W^{d,r}(\Omega; \mathbf{\Lambda}^{k_i-1})$, for any $1 \leq r < \infty$. This shows the claim.

Now using the bound on f and the fact that f is continuous since it is vectorially ext. quasiconvex, using Fatou's lemma we obtain,

$$\liminf_{\nu \rightarrow \infty} \int_{\Omega} \left[\alpha \left(1 + \sum_{i=1}^m G_i^u(\phi_i^\nu) \right) - f(\boldsymbol{\xi} + d\phi^\nu) \right] \geq \int_{\Omega} \left[\alpha \left(1 + \sum_{i=1}^m G_i^u(\phi_i) \right) - f(\boldsymbol{\xi} + d\phi) \right].$$

Now since $\{\phi_i^\nu\}$ is uniformly bounded in $W^{d,p_i}(\Omega; \Lambda^{k_i-1})$, using dominated convergence theorem we deduce,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} \left(1 + \sum_{i=1}^m G_i^u(\phi_i^\nu) \right) = \int_{\Omega} \left(1 + \sum_{i=1}^m G_i^u(\phi_i) \right).$$

Hence using vectorial ext. quasiconvexity of f , we deduce,

$$\int_{\Omega} f(\boldsymbol{\xi} + d\phi) \geq \limsup_{\nu \rightarrow \infty} \int_{\Omega} f(\boldsymbol{\xi} + d\phi^\nu) \geq \int_{\Omega} f(\boldsymbol{\xi}).$$

This proves the lemma. ■

We now generalize an elementary proposition from convex analysis in this setting. The proof is straightforward and is just a matter of iterating the argument in the proof of Proposition 2.32 in [25].

Proposition 4.18 *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i < \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \mathbf{\Lambda}^{\mathbf{k}} \rightarrow \mathbb{R}$ be separately convex and satisfy, for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbf{\Lambda}^{\mathbf{k}}$,*

$$|f(\boldsymbol{\xi})| \leq \alpha \left(1 + \sum_{i=1}^m |\xi_i|^{p_i} \right),$$

where $\alpha > 0$ is a constant. Then there exist constants $\beta_i > 0, i = 1, \dots, m$ such that

$$|f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta})| \leq \sum_{i=1}^m \beta_i \left(1 + \sum_{j=1}^m \left(|\xi_j|^{\frac{p_j}{p'_i}} + |\zeta_j|^{\frac{p_j}{p'_i}} \right) \right) |\xi_i - \zeta_i|,$$

for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m), \boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in \mathbf{\Lambda}^{\mathbf{k}}$, where p'_i is the Hölder conjugate of exponent of p_i .

Proof We know that for any convex function $g : \mathbb{R} \rightarrow \mathbb{R}$, we have, for every $\lambda > \mu > 0$ and for every $t \in \mathbb{R}$,

$$\frac{g(t \pm \mu) - g(t)}{\mu} \leq \frac{g(t \pm \lambda) - g(t)}{\lambda}.$$

The strategy for the proof is to use these inequalities for suitable choice of λ and μ , when all but one of the components of $\boldsymbol{\xi}$ is fixed. To this end, we define, for any $1 \leq i \leq m$, any $I \in \Lambda^{k_i}$,

$$\tilde{\boldsymbol{\xi}}^{i,I} := \left(\xi_1, \dots, \xi_{i-1}, \xi_i^I, \dots, \widehat{\xi_i^I}, \dots, \xi_i^{I_{N_i}}, \xi_{i+1} \dots \xi_m \right).$$

In words, $\tilde{\boldsymbol{\xi}}^{i,I}$ is the vector whose components are precisely all the components of $\boldsymbol{\xi}$ except ξ_i^I . Now let

$$g_i^I(t) := f(t, \tilde{\boldsymbol{\xi}}^{i,I}).$$

Choosing $\mu = \zeta_i^I - \xi_i^I$ and $\lambda = 1 + |\xi_i| + |\zeta_i| + \sum_{j \neq i} |\xi_j|^{\frac{p_j}{p'_i}}$, we obtain,

$$g(\zeta_i^I) - g(\xi_i^I) = g(\xi_i^I + \mu) - g(\xi_i^I) \leq \mu \frac{g(\xi_i^I + \lambda) - g(\xi_i^I)}{\lambda}$$

Using the growth conditions, this implies that there is a constant C such that,

$$\begin{aligned}
& \mu \frac{g(\xi_i^I + \lambda) - g(\xi_i^I)}{\lambda} \\
& \leq \frac{\alpha \left(1 + |\xi_i^I + \lambda|^{p_i} + \sum_{J \neq I} |\xi_i^J|^{p_i} + \sum_{j \neq i} |\xi_j|^{p_j} \right) + \alpha \left(1 + \sum_{j=1}^m |\xi_j|^{p_j} \right)}{\left(1 + |\xi_i| + |\zeta_i| + \sum_{j \neq i} |\xi_j|^{\frac{p_j}{p_i'}} \right)} |\xi_i^I - \zeta_i^I| \\
& \leq C \left(1 + |\xi_i|^{\frac{p_i}{p_i'}} + |\zeta_i|^{\frac{p_i}{p_i'}} + \sum_{j \neq i} |\xi_j|^{\frac{p_j}{p_i'}} \right) |\xi_i^I - \zeta_i^I|.
\end{aligned}$$

This gives,

$$g(\zeta_i^I) - g(\xi_i^I) \leq C \left(1 + |\xi_i|^{\frac{p_i}{p_i'}} + |\zeta_i|^{\frac{p_i}{p_i'}} + \sum_{j \neq i} |\xi_j|^{\frac{p_j}{p_i'}} \right) |\xi_i^I - \zeta_i^I|.$$

Exactly the same way, the same estimate can be derived for $g(\xi_i^I) - g(\zeta_i^I)$. Hence, we have,

$$|g(\zeta_i^I) - g(\xi_i^I)| \leq C \left(1 + |\xi_i|^{\frac{p_i}{p_i'}} + |\zeta_i|^{\frac{p_i}{p_i'}} + \sum_{j \neq i} |\xi_j|^{\frac{p_j}{p_i'}} \right) |\xi_i^I - \zeta_i^I|.$$

Our plan is to write $f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta})$ as sum of differences of functions, whereas in each such difference, only one component changes and the others are kept fixed. We plan to use the estimate above to each such difference. The only trouble is, the estimate is not symmetric with respect to the endpoints. When writing $f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta})$ as sum of differences of functions, the ‘fixed’ components will not always be fixed at their values at $\boldsymbol{\xi}$, but some components will be fixed at their values at $\boldsymbol{\xi}$ and some components at their values at $\boldsymbol{\zeta}$. So we can not really use precisely this estimate to all such differences. But that is easily rectified as the estimate above immediately yield the estimate,

$$|g(\zeta_i^I) - g(\xi_i^I)| \leq C \left(1 + |\xi_i|^{\frac{p_i}{p_i'}} + |\zeta_i|^{\frac{p_i}{p_i'}} + \sum_{j \neq i} \left(|\xi_j|^{\frac{p_j}{p_i'}} + |\zeta_j|^{\frac{p_j}{p_i'}} \right) \right) |\xi_i^I - \zeta_i^I|.$$

We can also get rid of the dependence of I on the right hand side completely, as this implies,

$$|g(\zeta_i^I) - g(\xi_i^I)| \leq C \left(1 + |\xi_i|^{\frac{p_i}{p_i'}} + |\zeta_i|^{\frac{p_i}{p_i'}} + \sum_{j \neq i} \left(|\xi_j|^{\frac{p_j}{p_i'}} + |\zeta_j|^{\frac{p_j}{p_i'}} \right) \right) |\xi_i - \zeta_i|.$$

This estimate now is true for all such differences. Stitching the argument together, this gives the desired inequality and finishes the proof. ■

Now we generalize this proposition to cover the case where some of the p_i s can be ∞ as well.

Proposition 4.19 *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. Let $0 \leq r \leq m$ be an integer. Let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i < \infty$ for all $1 \leq i \leq r$ and $p_{r+1} = \dots = p_m = \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \boldsymbol{\Lambda}^{\mathbf{k}} \rightarrow \mathbb{R}$ be separately convex and satisfy, for*

every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}}$,

$$|f(\boldsymbol{\xi})| \leq \alpha \left(1 + \sum_{i=1}^r |\xi_i|^{p_i} + \sum_{i=r+1}^m \eta_i(|\xi_i|) \right),$$

where $\alpha > 0$ is a constant and η_i s are some nonnegative, continuous, increasing functions. Let

$$Q := [-C, C]^{\sum_{i=r+1}^m \binom{n}{k_i}} \subset \prod_{i=r+1}^m \Lambda^{k_i}$$

be a cube and define

$$K := \Lambda^{k_1} \times \dots \times \Lambda^{k_r} \times Q.$$

Then there exist constants $\beta_i = \beta_i(K) > 0, i = 1, \dots, m$ such that

$$\begin{aligned} |f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta})| &\leq \sum_{i=1}^r \beta_i \left(1 + \sum_{j=1}^r \left(|\xi_j|^{\frac{p_j}{p'_i}} + |\zeta_j|^{\frac{p_j}{p'_i}} \right) \right) |\xi_i - \zeta_i| \\ &\quad + \sum_{i=r+1}^m \beta_i \left(1 + \sum_{j=1}^r (|\xi_j|^{p_j} + |\zeta_j|^{p_j}) \right) |\xi_i - \zeta_i|, \end{aligned} \quad (4.17)$$

for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m), \boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m) \in K$, where p'_i is the Hölder conjugate of exponent of p_i .

Remark 4.20 1. Clearly, when $r = m$, the last term in the inequality (4.17) is not present.

2. Of course, the assumption on the naming of the variable is not a restriction at all, since we can always relabel the variables.

Proof We write,

$$f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta}) = f(\boldsymbol{\xi}) - f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m) + f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m) - f(\boldsymbol{\zeta}).$$

Hence we have,

$$|f(\boldsymbol{\xi}) - f(\boldsymbol{\zeta})| \leq |f(\boldsymbol{\xi}) - f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m)| + |f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m) - f(\boldsymbol{\zeta})|. \quad (4.18)$$

Now since $|\xi_i| \leq C$ and η_i s are continuous for all $r+1 \leq i \leq m$, the function

$$h(\gamma_1, \dots, \gamma_r) = f(\gamma_1, \dots, \gamma_r, \xi_{r+1}, \dots, \xi_m)$$

satisfies the growth condition

$$|h(\gamma_1, \dots, \gamma_r)| \leq \rho(K) \left(1 + \sum_{i=1}^r |\gamma_i|^{p_i} \right),$$

where the constant ρ depends on both α and the set K , or more precisely on the bound C . Hence, using proposition 4.18 on h , we obtain,

$$|h(\xi_1, \dots, \xi_r) - h(\zeta_1, \dots, \zeta_r)| \leq \sum_{i=1}^r \beta_i \left(1 + \sum_{j=1}^r \left(|\xi_j|^{\frac{p_j}{p'_i}} + |\zeta_j|^{\frac{p_j}{p'_i}} \right) \right) |\xi_i - \zeta_i|.$$

This gives,

$$|f(\xi) - f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m)| \leq \sum_{i=1}^r \beta_i \left(1 + \sum_{j=1}^r \left(|\xi_j|^{\frac{p_j}{p'_i}} + |\zeta_j|^{\frac{p_j}{p'_i}} \right) \right) |\xi_i - \zeta_i|.$$

Hence, our proof will be finished if we show that

$$|f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m) - f(\xi)| \leq \sum_{i=r+1}^m \beta_i \left(1 + \sum_{j=1}^r (|\xi_j|^{p_j} + |\zeta_j|^{p_j}) \right) |\xi_i - \zeta_i|.$$

To this end, we note once again that for any convex function $g : \mathbb{R} \rightarrow \mathbb{R}$, we have, for any $x, y \in [-C, C]$,

$$|g(x) - g(y)| \leq \frac{M - m}{2C} |x - y|,$$

where $M = \max_{|t| \leq 4C} g(t)$ and $m = \min_{|t| \leq 4C} g(t)$. This implies the estimate

$$|g(x) - g(y)| \leq \frac{M'}{C} |x - y|,$$

where $M' = \max_{|t| \leq 4C} |g(t)|$. Using separate convexity and writing as a sum of differences of functions, whereas in each such difference, only one component changes and the others are kept fixed, this immediately generalizes to the estimate,

$$|G(x) - G(y)| \leq \sum_{i=r+1}^m \frac{c\tilde{M}}{C} |x_i - y_i|,$$

for every $x, y \in Q$, for every separately convex function $G : \prod_{i=r+1}^m \Lambda^{k_i} \rightarrow \mathbb{R}$, where $c > 0$ is a constant and $\tilde{M} = \max_{t \in 4Q} |G(t)|$ is the maximum of $|G|$ in the cube

$$4Q := [-4C, 4C]^{\sum_{i=r+1}^m \binom{n}{k_i}} \subset \prod_{i=r+1}^m \Lambda^{k_i}.$$

Setting

$$G(\gamma_{r+1}, \dots, \gamma_m) = f(\zeta_1, \dots, \zeta_r, \gamma_{r+1}, \dots, \gamma_m)$$

and using this estimate for G , we obtain, by the growth condition on f ,

$$|G(\xi_{r+1}, \dots, \xi_m) - G(\zeta_{r+1}, \dots, \zeta_m)| \leq \sum_{i=r+1}^m \frac{\beta(K) \left(1 + \sum_{j=1}^r |\zeta_j|^{p_j} \right)}{C} |x_i - y_i|.$$

This implies,

$$|G(\xi_{r+1}, \dots, \xi_m) - G(\zeta_{r+1}, \dots, \zeta_m)| \leq \sum_{i=r+1}^m \frac{\beta(K) \left(1 + \sum_{j=1}^r (|\xi_j|^{p_j} + |\zeta_j|^{p_j})\right)}{C} |x_i - y_i|.$$

This immediately implies the estimate

$$|f(\zeta_1, \dots, \zeta_r, \xi_{r+1}, \dots, \xi_m) - f(\zeta)| \leq \sum_{i=r+1}^m \beta_i \left(1 + \sum_{j=1}^r (|\xi_j|^{p_j} + |\zeta_j|^{p_j})\right) |\xi_i - \zeta_i|$$

and finishes the proof. ■

Now we are in a position to prove the semicontinuity result. We start with a lemma which is essentially about changing the boundary values of a sequence. In classical calculus of variations, such a lemma is well-known (see Acerbi-Fusco[1], Marcellini[48], Meyers[49], Morrey[52], [53] etc.).

Lemma 4.21 *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$. Let $D \subset \mathbb{R}^n$ be a cube parallel to the axes. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbf{\Lambda}^{\mathbf{k}}$. Let $f : \mathbf{\Lambda}^{\mathbf{k}} \rightarrow \mathbb{R}$ be vectorially ext. quasiconvex satisfying the growth condition $(C_{\mathbf{p}})$. Let*

$$\phi^\nu \rightharpoonup 0 \quad \text{in } W^{d, \mathbf{p}}(D; \mathbf{\Lambda}^{\mathbf{k}-1}) \quad (* \text{ if } p_i = \infty).$$

Then

$$\liminf_{\nu \rightarrow \infty} \int_D f(\boldsymbol{\xi} + d\phi^\nu) \geq f(\boldsymbol{\xi}) \text{meas}(D).$$

Proof Let $D^0 \subset\subset D$ be a cube having sides parallel to the axes and let

$$R := \frac{1}{2} \text{dist}(D^0, \partial D).$$

Let M be an integer and let $D^0 \subset D^\mu \subset D$ be a family of cubes each having sides parallel to the axes, $1 \leq \mu \leq M$ integers, be such that

$$\text{dist}(D^0, \partial D^\mu) = \frac{\mu}{M}, \quad 1 \leq \mu \leq M.$$

We then choose $\theta_\mu \in C_c^\infty(D)$, $1 \leq \mu \leq M$, such that

$$0 \leq \theta_\mu \leq 1, \quad |D\theta_\mu| \leq \frac{aM}{R}, \quad \theta_\mu = \begin{cases} 1 & \text{if } x \in D^{\mu-1} \\ 0 & \text{if } x \in D - D^{\mu-1}, \end{cases}$$

where $a > 0$ is a constant. Let

$$\omega_\mu^\nu = \theta_\mu \phi^\nu.$$

Since $\omega_\mu^\nu \in W_T^{d,p}(\Omega; \Lambda^{k-1})$, by lemma 4.17, we obtain,

$$\begin{aligned} \int_D f(\xi) &\leq \int_D f(\xi + d\omega_\mu^\nu(x)) \\ &= \int_{D-D^\mu} f(\xi) + \int_{D^\mu-D^{\mu-1}} f(\xi + d\omega_\mu^\nu(x)) + \int_{D^{\mu-1}} f(\xi + d\phi^\nu(x)). \end{aligned}$$

This implies,

$$\int_{D^\mu} f(\xi) \leq \int_{D^\mu-D^{\mu-1}} f(\xi + d\omega_\mu^\nu(x)) + \int_{D^{\mu-1}} f(\xi + d\phi^\nu(x)).$$

Rewriting, we obtain,

$$\begin{aligned} \int_{D^\mu} f(\xi) &\leq \int_D f(\xi + d\phi^\nu(x)) - \int_{D-D^{\mu-1}} f(\xi + d\phi^\nu(x)) + \int_{D^\mu-D^{\mu-1}} f(\xi + d\omega_\mu^\nu(x)) \\ &= \int_D f(\xi + d\phi^\nu(x)) + I_1 + I_2. \end{aligned} \quad (4.19)$$

Now we estimate I_1 and I_2 .

Estimation of I_1 : Using (\mathcal{C}_p) , we have,

$$\begin{aligned} |I_1| &\leq \alpha \int_{D-D^{\mu-1}} \left(1 + \sum_{i=1}^m G_i^l(\xi + d\phi^\nu) \right) \\ &\leq \alpha' \int_{D-D^0} \left(1 + \sum_{i=1}^m \gamma_i |\xi_i|^{\tilde{q}_i} + \sum_{i=1}^m \gamma_i |d\phi_i^\nu|^{\tilde{q}_i} \right), \end{aligned} \quad (4.20)$$

where $\alpha', \gamma_i > 0$ are constants and the powers \tilde{q}_i are given by,

$$\tilde{q}_i = \begin{cases} 1, & \text{if } p_i = 1, \\ q_i, & \text{the powers in the lower bound in } (\mathcal{C}_p) \text{ if } 1 < p_i < \infty, \\ 0, & \text{if } p_i = \infty. \end{cases}$$

The validity of such an estimate is obvious for the terms for which $1 \leq p_i < \infty$. For the terms where $p_i = \infty$ follows from the fact that since $|\xi_i + d\phi_i^\nu|$ is uniformly bounded in L^∞ and η_i are continuous, we have the estimate

$$\eta_i(|\xi_i + d\phi_i^\nu|) \leq C'.$$

We proceed from (4.20). The terms for which $p_i = \tilde{q}_i = 1$ can be made as small as we please by choosing R small enough by equiintegrability of the sequence $\{d\phi_i^\nu\}$. For the other terms where i is such that $1 < p_i < \infty$, we use the fact that $\tilde{q}_i = q_i < p_i$ and hence using Hölder inequality, we obtain,

$$\int_{D-D^0} |d\phi_i^\nu|^{\tilde{q}_i} \leq \left(\int_{D-D^0} |d\phi_i^\nu|^{p_i} \right)^{\frac{q_i}{p_i}} (\text{meas}(D - D_0))^{\frac{p_i - q_i}{p_i}}.$$

Hence, by choosing R sufficiently small, these terms can be made arbitrarily small as well. Combining all these, we get, for any fixed given $\epsilon > 0$, we can obtain, for R small enough,

$$|I_1| \leq \epsilon. \quad (4.21)$$

Estimation of I_2 : Using (\mathcal{C}_p) , we have,

$$|I_2| \leq \alpha \int_{D^\mu - D^{\mu-1}} \left(1 + \sum_{i=1}^m G_i^u(\boldsymbol{\xi} + d\boldsymbol{\omega}_\mu^\nu) \right),$$

Note that

$$d\omega_{\mu,i}^\nu = \theta_\mu d\phi_i^\nu + \nabla \theta_\mu \wedge \phi_i^\nu,$$

for all $1 \leq i \leq m$. Using this and the uniform bounds, we deduce,

$$|I_2| \leq \alpha' \int_{D^\mu - D^{\mu-1}} \left(1 + \sum_{p_i \neq \infty} \left(\gamma_i |\xi_i|^{p_i} + \gamma_i' |d\phi_i^\nu|^{p_i} + \gamma_i'' \left(\frac{aM}{R} \right) |\phi_i^\nu|^{p_i} \right) \right). \quad (4.22)$$

Now we simply plan to sum these estimates with μ running from 1 to M , noting that the domain of integration on the right hand side of the last estimate telescopes. This trick of using the telescoping sum to avoid concentration was first used by De Giorgi [28] (see also Marcellini [48]), in the classical calculus of variations. So, returning back to (4.19) and adding from $\mu = 1$ to M and dividing by M , we obtain,

$$\begin{aligned} & \int_D f(\boldsymbol{\xi} + d\boldsymbol{\phi}^\nu(x)) - \frac{f(\boldsymbol{\xi})}{M} \sum_{\mu=1}^M \text{meas}(D^\mu) \\ & \geq -\epsilon - \frac{\alpha'}{M} \int_{D^\mu - D^0} \left(1 + \sum_{p_i \neq \infty} \left(\gamma_i |\xi_i|^{p_i} + \gamma_i' |d\phi_i^\nu|^{p_i} + \gamma_i'' \left(\frac{aM}{R} \right) |\phi_i^\nu|^{p_i} \right) \right) \\ & \geq -\epsilon - \frac{\alpha''}{M} \end{aligned} \quad (4.23)$$

Now since

$$\text{meas}(D_0) \leq \frac{1}{M} \sum_{\mu=1}^M \text{meas}(D^\mu) \leq \text{meas}(D),$$

and ϵ and D_0 is arbitrary, taking $M \rightarrow \infty$, we obtain,

$$\liminf_{\nu \rightarrow \infty} \int_D f(\boldsymbol{\xi} + d\boldsymbol{\phi}^\nu(x)) \geq f(\boldsymbol{\xi}) \text{meas}(D).$$

This completes the proof. \blacksquare

Theorem 4.22 *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. Let $0 \leq r \leq m$ be an integer. $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i < \infty$ for all $1 \leq i \leq r$ and $p_{r+1} = \dots = p_m = \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \mathbf{\Lambda}^{\mathbf{k}} \rightarrow \mathbb{R}$ be vectorially ext. quasiconvex, satisfying*

the growth condition (\mathcal{C}_p) . Let $I : W^{d,p}(\Omega; \Lambda^{k-1}) \rightarrow \mathbb{R}$ defined by

$$I(\omega) := \int_{\Omega} f(d\omega(x)) \, dx, \text{ for all } \omega \in W^{d,p}(\Omega; \Lambda^{k-1}).$$

Then I is weakly lower semicontinuous in $W^{d,p}(\Omega; \Lambda^{k-1})$.

Proof We need to show that

$$\liminf_{\nu \rightarrow \infty} I(\omega^\nu) \geq I(\omega),$$

for any sequence

$$\omega^\nu \rightharpoonup \omega \quad \text{in } W^{d,p}(\Omega; \Lambda^{k-1}) \quad (\overset{*}{\rightharpoonup} \text{ if } p_i = \infty).$$

We divide the proof into several steps.

Step 1 First we show that it is enough to prove the theorem under the additional hypotheses that $|d\omega_i^\nu|^{p_i}$ is equiintegrable for every $1 \leq i \leq r$. Suppose we have shown the theorem with this additional assumption. Then for any sequence

$$\omega^\nu \rightharpoonup \omega \quad \text{in } W^{d,p}(\Omega; \Lambda^{k-1}),$$

we first restrict our attention to a subsequence, still denoted by $\{\omega^\nu\}$ such that the limit inferior is realized, i.e

$$L := \liminf_{\nu \rightarrow \infty} \int_{\Omega} f(d\omega^\nu(x)) \, dx = \lim_{\nu \rightarrow \infty} \int_{\Omega} f(d\omega^\nu(x)) \, dx.$$

Now we use a decomposition lemma in calculus of variations (cf. Lemma 2.15 in [29]) to find, passing to a subsequence if necessary, a sequence $\{v_i^\nu\} \subset L^{p_i}$ such that $\{|v_i^\nu|^{p_i}\}$ is equiintegrable and

$$v_i^\nu \rightharpoonup d\omega_i \text{ in } L^{p_i}(\Omega, \Lambda^{k_i})$$

and

$$\lim_{\nu \rightarrow \infty} \text{meas } \Omega_\nu = 0,$$

where

$$\Omega_\nu := \{x \in \Omega : v_i^\nu(x) \neq d\omega_i^\nu(x)\},$$

for all $1 \leq i \leq r$ with $p_i > 1$. Note also that if $p_i = 1$, we can take $v_i^\nu = d\omega_i^\nu$.

Now, we have, using (\mathcal{C}_p) ,

$$\begin{aligned} \int_{\Omega} f(d\omega^\nu(x)) \, dx &\geq \int_{\Omega \setminus \Omega_\nu} f(v_1^\nu(x), \dots, v_r^\nu(x), d\omega_{r+1}^\nu(x), \dots, d\omega_{r+1}^\nu(x)) \, dx \\ &\quad - \alpha \int_{\Omega_\nu} \left(C + \sum_{i=1}^r |d\omega_i^\nu|^{\tilde{q}_i} \right), \end{aligned}$$

where C is a positive constant, depending on the uniform L^∞ bounds of $\{d\omega_i^\nu\}$ and η_i s in (\mathcal{C}_p) , for all $r+1 \leq i \leq m$ and $\tilde{q}_i = q_i$, as given in (\mathcal{C}_p) , if $p_i > 1$ and $\tilde{q}_i = 1$ if $p_i = 1$ for any $1 \leq i \leq m$.

Using (\mathcal{C}_p) again, we obtain,

$$\int_{\Omega} f(\mathbf{d}\omega^\nu(x)) \geq \int_{\Omega} f(v_1^\nu, \dots, v_r^\nu, d\omega_{r+1}^\nu, \dots, d\omega_{r+1}^\nu) - \alpha \int_{\Omega_\nu} \left(C + \sum_{i=1}^r (|d\omega_i^\nu|^{\tilde{q}_i} + |v_i^\nu|^{p_i}) \right).$$

Now we have $\lim_{\nu \rightarrow \infty} \text{meas } \Omega_\nu = 0$, $\{|v_i^\nu|^{p_i}\}$ is equiintegrable by construction and $\{|d\omega_i^\nu|^{\tilde{q}_i}\}$ is equiintegrable since $\tilde{q}_i = q_i < p_i$ if $p_i > 1$ and $\tilde{q}_i = 1$ if $p_i = 1$. Using these facts, we obtain,

$$L = \lim_{\nu \rightarrow \infty} \int_{\Omega} f(\mathbf{d}\omega^\nu(x)) dx \geq \liminf_{\nu \rightarrow \infty} \int_{\Omega} f(v_1^\nu, \dots, v_r^\nu, d\omega_{r+1}^\nu, \dots, d\omega_{r+1}^\nu) \geq \int_{\Omega} f(\mathbf{d}\omega(x)) dx,$$

by hypotheses. This proves our claim.

Step 2 Now by Step 1, we can assume, in addition that $|d\omega_i^\nu|^{p_j}$ is equiintegrable for every $1 \leq i \leq r$. Now we approximate Ω by a union of cubes D_s with sides parallel to the axes and whose edge length is $\frac{1}{h}$, where h is an integer. We denote this union by H_h and choose h large enough such that

$$\text{meas}(\Omega - H_h) \leq \delta \quad \text{where } H_h := \bigcup D_s.$$

Also, we define the average of $d\omega_i$ over each of the cubes D_s to be,

$$\xi_s^i := \frac{1}{\text{meas}(D_s)} \int_{D_s} d\omega_i \in \Lambda^{k_i}.$$

Also, let $\xi_s := (\xi_s^1, \dots, \xi_s^m)$ and $\xi(x) := \xi_s \chi_{D_s}(x)$ for every $x \in H_h$. Since as the size of the cubes shrink to zero, $d\omega_i$ converges to ξ_i in $L^{p_i}(\Omega; \Lambda^{k_i})$ for each $1 \leq i \leq r$, we obtain, by choosing h large enough,

$$\left(\sum_s \int_{D_s} |d\omega_i - \xi_s^i|^{p_i} \right)^{\frac{1}{p_i}} \leq C_1 \epsilon, \quad (4.24)$$

for every $1 \leq i \leq r$. Also, by the same argument, we obtain, by choosing h large enough,

$$\sum_s \int_{D_s} |d\omega_i - \xi_s^i| \leq C_2 \epsilon, \quad (4.25)$$

for every $r+1 \leq i \leq m$.

Now consider

$$\begin{aligned} I(\omega^\nu) - I(\omega) &= \int_{\Omega} [f(\mathbf{d}\omega^\nu(x)) - f(\mathbf{d}\omega(x))] dx \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_{\Omega-H_h} [f(\mathbf{d}\omega^\nu(x)) - f(\mathbf{d}\omega(x))] \, dx, \\
I_2 &:= \sum_s \int_{D_s} [f(\mathbf{d}\omega + (\mathbf{d}\omega^\nu - \mathbf{d}\omega)) - f(\xi_s + (\mathbf{d}\omega^\nu - \mathbf{d}\omega))] \, dx, \\
I_3 &:= \sum_s \int_{D_s} [f(\xi_s + (\mathbf{d}\omega^\nu - \mathbf{d}\omega)) - f(\xi_s)] \, dx, \\
I_4 &:= \sum_s \int_{D_s} [f(\xi_s) - f(\mathbf{d}\omega)] \, dx.
\end{aligned}$$

Now we estimate I_1, I_2 and I_4 .

Estimation of I_1 : Using the growth condition (\mathcal{C}_p) , we have,

$$\begin{aligned}
I_1 &\geq - \int_{\Omega-H_h} \left[\alpha \left(1 + \sum_{i=1}^m G_i^l(\mathbf{d}\omega^\nu) \right) + f(\mathbf{d}\omega) \right] \\
&\geq - \int_{\Omega-H_h} (\alpha + f(\mathbf{d}\omega)) - \alpha' \int_{\Omega-H_h} \sum_{i=1}^m \gamma_i |\mathbf{d}\omega_i^\nu|^{\tilde{q}_i}, \tag{4.26}
\end{aligned}$$

where $\alpha', \gamma_i > 0$ are constants and the powers \tilde{q}_i are given by,

$$\tilde{q}_i = \begin{cases} 1, & \text{if } p_i = 1, \\ q_i, & \text{the powers in the lower bound in } (\mathcal{C}_p) \text{ if } 1 < p_i < \infty, \\ 0, & \text{if } p_i = \infty. \end{cases}$$

The validity of such an estimate is obvious for the terms for which $1 \leq p_i < \infty$. For the terms where $p_i = \infty$ follows from the fact that since $|\mathbf{d}\omega_i^\nu|$ is uniformly bounded in L^∞ and η_i are continuous, we have the estimate

$$\eta_i(|\mathbf{d}\omega_i^\nu|) \leq C'.$$

Now we proceed from (4.26). The terms for which $p_i = \tilde{q}_i = 1$ can be made as small as we please, uniformly in ν , by choosing δ small enough by equiintegrability of the sequence $\{\mathbf{d}\omega_i^\nu\}$. For the other terms where i is such that $1 < p_i < \infty$, we use the fact that $\tilde{q}_i = q_i < p_i$ and hence using Hölder inequality, we obtain,

$$\int_{\Omega-H_h} |\mathbf{d}\omega_i^\nu|^{\tilde{q}_i} \leq \left(\int_{\Omega-H_h} |\mathbf{d}\omega_i^\nu|^{p_i} \right)^{\frac{q_i}{p_i}} (\text{meas}(\Omega - H_h))^{\frac{p_i - q_i}{p_i}}.$$

Hence, by choosing δ sufficiently small, these terms can be made arbitrarily small uniformly in ν as well. Combining all these, we get, for any fixed given $\epsilon > 0$, choosing δ small enough, we obtain

$$I_1 \geq -C_{12}\epsilon \tag{4.27}$$

uniformly in ν .

Estimation of I_2 : Since f is vectorially ext. quasiconvex, it is separately convex and since both $\{\mathbf{d}\omega_i + (\mathbf{d}\omega_i^\nu - \mathbf{d}\omega_i)\}$ and $\{\xi_s^i + (\mathbf{d}\omega_i^\nu - \mathbf{d}\omega_i)\}$ is uniformly bounded in $L^\infty(\Omega; \Lambda^{k_i})$ for every

$r + 1 \leq i \leq m$, using proposition 4.19, we have,

$$|I_2| \leq \sum_s \int_{D_s} \sum_{i=1}^r \beta_i \left(1 + \sum_{j=1}^r \left(|d\omega_j + (d\omega_j^\nu - d\omega_j)|^{\frac{p_j}{p_i}} + |\xi_s^j + (d\omega_j^\nu - d\omega_j)|^{\frac{p_j}{p_i}} \right) \right) |d\omega_i - \xi_s^i| \\ + \sum_s \int_{D_s} \sum_{i=r+1}^m \beta_i \left(1 + \sum_{j=1}^r (|d\omega_j + (d\omega_j^\nu - d\omega_j)|^{p_j} + |\xi_s^j + (d\omega_j^\nu - d\omega_j)|^{p_j}) \right) |d\omega_i - \xi_s^i|$$

The terms in the first sum can be easily estimated by using Hölder inequality and the estimate (4.24). So we concentrate on the second sum.

We have,

$$\sum_s \int_{D_s} \sum_{i=r+1}^m \beta_i \left(1 + \sum_{j=1}^r (|d\omega_j + (d\omega_j^\nu - d\omega_j)|^{p_j} + |\xi_s^j + (d\omega_j^\nu - d\omega_j)|^{p_j}) \right) |d\omega_i - \xi_s^i| \\ \leq \sum_s \int_{D_s} \sum_{i=r+1}^m \tilde{\beta}_i \left(1 + \sum_{j=1}^r (|d\omega_j^\nu|^{p_j} + |d\omega_j - \xi_s^j|^{p_j}) \right) |d\omega_i - \xi_s^i|,$$

for some positive constants $\tilde{\beta}_i$ s.

Now the terms of the form

$$\sum_s \int_{D_s} \tilde{\beta}_i |d\omega_i - \xi_s^i|$$

can be easily estimated using the estimate (4.25). For the other terms, for clarity of presentation, we fix $r + 1 \leq i \leq m$ and $1 \leq j \leq r$. For such i, j fixed, we obtain,

$$\sum_s \int_{D_s} \tilde{\beta}_i |d\omega_j - \xi_s^j|^{p_j} |d\omega_i - \xi_s^i| \leq 2\tilde{\beta}_i \|d\omega_i\|_{L^\infty(\Omega)} \sum_s \int_{D_s} |d\omega_j - \xi_s^j|^{p_j}, \quad (4.28)$$

since $|d\omega_i - \xi_s^i| \leq 2\|d\omega_i\|_{L^\infty(\Omega)}$ for any s . Using the estimate (4.24), this shows that these terms can be made as small as we please by choosing h large enough. Now estimate for the terms of the type

$$\sum_s \int_{D_s} \tilde{\beta}_i |d\omega_j^\nu|^{p_j} |d\omega_i - \xi_s^i|$$

is a bit more involved. Once again, we fix $r + 1 \leq i \leq m$ and $1 \leq j \leq r$. Since $\{|d\omega_j^\nu|^{p_j}\}$ is uniformly bounded in L^1 and is equiintegrable, we know,

$$\lim_{M \rightarrow \infty} \sup_\nu \int_{\Omega \cap \{|d\omega_j^\nu|^{p_j} > M\}} |d\omega_j^\nu|^{p_j} = 0.$$

This implies, for any $\epsilon > 0$, there exists $M = M(\epsilon)$ such that

$$\int_{\Omega \cap \{|d\omega_j^\nu|^{p_j} > M\}} |d\omega_j^\nu|^{p_j} < \frac{\epsilon}{2\tilde{\beta}_i \|d\omega_i\|_{L^\infty(\Omega)}} \text{ for all } \nu.$$

Thus, we have,

$$\begin{aligned}
& \sum_s \int_{D_s} \tilde{\beta}_i |d\omega_j^\nu|^{p_j} |d\omega_i - \xi_s^i| \\
&= \int_{H_h \cap \{|d\omega_j^\nu|^{p_j} > M\}} \tilde{\beta}_i |d\omega_j^\nu|^{p_j} |d\omega_i - \xi_s^i| + \int_{H_h \cap \{|d\omega_j^\nu|^{p_j} \leq M\}} \tilde{\beta}_i |d\omega_j^\nu|^{p_j} |d\omega_i - \xi_s^i| \\
&\leq \epsilon + \tilde{\beta}_i M \sum_s \int_{D_s} |d\omega_i - \xi_s^i|.
\end{aligned}$$

By (4.25), we can choose h large enough such that

$$\sum_s \int_{D_s} |d\omega_i - \xi_s^i| \leq \frac{\epsilon}{\tilde{\beta}_i M}.$$

Combining, by choosing h large enough, we deduce,

$$I_2 \geq -C_{22}\epsilon \quad (4.29)$$

uniformly in ν .

Estimation of I_4 : This estimate is similar but simpler than that of I_2 . Using the same arguments as above and using proposition 4.19, we obtain, by choosing h large enough,

$$I_2 \geq -C_{42}\epsilon \quad (4.30)$$

uniformly in ν .

Now we finish the proof. Using the estimates (4.27), (4.29) and (4.30) and taking the limit $\nu \rightarrow \infty$, we obtain,

$$\liminf_{\nu \rightarrow \infty} I(\omega^\nu) - I(\omega) \geq -(C_{12} + C_{22} + C_{42})\epsilon + \sum_s \liminf_{\nu \rightarrow \infty} \int_{D_s} [f(\xi_s + (d\omega^\nu - d\omega)) - f(\xi_s)] dx. \quad (4.31)$$

Since

$$d\omega^\nu - d\omega \rightharpoonup 0 \quad \text{in } W^{d,p}(D_s; \Lambda^{k-1})$$

for every s , we have, by lemma 4.21

$$\liminf_{\nu \rightarrow \infty} \int_{D_s} f(\xi_s + (d\omega^\nu - d\omega)) dx \geq \int_{D_s} f(\xi_s) \quad \text{for every } s.$$

Combining this with (4.31) and the fact that ϵ is arbitrary, we have finished the proof of the theorem. ■

Lower semicontinuity for quasiconvex functions with dependence on x

We start by defining the growth conditions that we need.

Definition 4.23 (Growth condition) Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $f : \Omega \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function.

f is said to be of growth $(\mathcal{C}_{\mathbf{p}}^x)$, if, for almost every $x \in \Omega$ and for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}}$, f satisfies,

$$-\beta(x) - \sum_{i=1}^m G_i^l(\xi_i) \leq f(x, \boldsymbol{\xi}) \leq \beta(x) + \sum_{i=1}^m G_i^u(\xi_i), \quad (\mathcal{C}_{\mathbf{p}}^x)$$

where $\beta \in L^1(\Omega)$ is nonnegative and the functions G_i^l s in the lower bound and the functions G_i^u s in the upper bound has the following form:

- If $p_i = 1$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$\begin{aligned} G_i^l(\xi_i) &= \alpha_i |\xi_i|^{q_i} \\ &\text{and} \\ G_i^u(\xi_i) &= g_i(x) |\xi_i|^{p_i}, \end{aligned}$$

for some $1 \leq q_i < p_i$ and for some constant $\alpha_i \geq 0$ and some non-negative measurable function g_i .

- If $p_i = \infty$, then,

$$G_i^l(\xi_i) = G_i^u(\xi_i) = \eta_i(|\xi_i|).$$

for some nonnegative, continuous, increasing function η_i .

Now we are ready to prove the semicontinuity result for functionals with explicit dependence on x , but we first prove the result in a simplified setting.

Theorem 4.24 Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. Let $0 \leq r \leq m$ be an integer. $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i < \infty$ for all $1 \leq i \leq r$ and $p_{r+1} = \dots = p_m = \infty$. Let $\Omega \subset \mathbb{R}^n$ be an open cube with sides parallel to the axes. Let $f : \Omega \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function, satisfying, for almost every $x \in \Omega$ and for every $\boldsymbol{\xi} \in \Lambda^{\mathbf{k}}$,

$$-\sum_{\substack{i \\ p_i=1}} \alpha_i |\xi_i| \leq f(x, \boldsymbol{\xi}) \leq \beta(x) + \sum_{i=1}^r \alpha_i |\xi_i|^{p_i} + \sum_{i=r+1}^m \eta_i(|\xi_i|), \quad (\mathcal{C}_{\mathbf{p}}^{x'})$$

for some nonnegative $\beta \in L^1(\Omega)$, where $\alpha_i \geq 0$ for all $1 \leq i \leq r$ are constants and η_i s are some nonnegative, continuous, increasing function for each $r+1 \leq i \leq m$. Also, let $\boldsymbol{\xi} \mapsto f(x, \boldsymbol{\xi})$ is

vectorially ext. quasiconvex for a.e $x \in \Omega$. Let $I : W^{d,p}(\Omega; \mathbf{\Lambda}^{k-1}) \rightarrow \mathbb{R}$ defined by

$$I(\omega) := \int_{\Omega} f(x, d\omega(x)) dx, \text{ for all } \omega \in W^{d,p}(\Omega; \mathbf{\Lambda}^{k-1}).$$

Let

$$\omega^\nu \rightharpoonup \omega \quad \text{in } W^{d,p}(\Omega; \mathbf{\Lambda}^{k-1}) \quad (\xrightarrow{*} \text{ if } p_i = \infty),$$

with $\{|d\omega_i^\nu|^{p_i}\}$ is equiintegrable for every $1 \leq i \leq r$. Then

$$\liminf_{\nu \rightarrow \infty} I(\omega^\nu) \geq I(\omega).$$

Proof The strategy is to freeze the points and then use Theorem 4.22.

Step 1 Since $\{|d\omega_i^\nu|^{p_i}\}$ is uniformly bounded in L^1 for every $1 \leq i \leq r$, then for every $\epsilon > 0$, there exist constants $M_\epsilon^i \geq 1$, independent of ν , such that if

$$K_{\epsilon,\nu}^i := \{x \in \Omega : |d\omega_i^\nu|^{p_i} \text{ or } |d\omega_i|^{p_i} \geq M_\epsilon^i\},$$

then

$$\text{meas } K_{\epsilon,\nu}^i < \frac{\epsilon}{r},$$

for every $1 \leq i \leq r$ and for every ν .

We define

$$K_{\epsilon,\nu} := \bigcup_{i=1}^r K_{\epsilon,\nu}^i \quad \text{and} \quad \Omega_\epsilon := \Omega \setminus K_{\epsilon,\nu}.$$

Also, $\{\|d\omega_i^\nu\|_{L^\infty}\}$ is uniformly bounded for every $r+1 \leq i \leq m$, i.e there exist constants $\gamma_i > 0$ such that

$$\|d\omega_i^\nu\|_{L^\infty} \leq \gamma_i \quad \text{for all } \nu,$$

for all $r+1 \leq i \leq m$. Define

$$k := \sum_{i=r+1}^m \eta_i(\gamma_i).$$

Since $\beta \in L^1(\Omega)$ and nonnegative, given any $\epsilon > 0$, we can find $M_\epsilon^\beta \leq 1$ such that if

$$E_\epsilon := \{x \in \Omega : \beta(x) \leq M_\epsilon^\beta\}$$

then

$$\text{meas}(\Omega \setminus E_\epsilon) \leq \frac{\epsilon}{k}, \quad \int_{\Omega \setminus E_\epsilon} \beta(x) dx < \epsilon,$$

and, in particular,

$$M_\epsilon^\beta \text{meas}(\Omega \setminus E_\epsilon) < \epsilon.$$

Now by the Scorza-Dragoni theorem (cf. theorem 3.8 in [25]), we find a compact set $K_\epsilon \subset \Omega_\epsilon$ with

$$\text{meas}(\Omega_\epsilon \setminus K_\epsilon) < \epsilon$$

such that $f : K_\epsilon \times S_\epsilon$ is continuous, where

$$S_\epsilon := \{\boldsymbol{\xi} \in \mathbf{\Lambda}^{\mathbf{k}} : |\xi|^{p_i} < M_\epsilon^i \text{ for all } 1 \leq i \leq r, |\xi| < \gamma_i \text{ for all } r+1 \leq i \leq m\}.$$

Step 2 Now we divide Ω into a finite union of cubes D_s of side length $\frac{1}{h}$. Choosing h such that the edge length of the cube Ω is an integral multiple of $1/h$, we have,

$$\text{meas} \left(\Omega \setminus \bigcup_s D_s \right) = 0.$$

Estimation of $\int_\Omega f(x, d\boldsymbol{\omega})$:

We fix $x_s \in D_s$ and obtain,

$$\begin{aligned} \int_\Omega f(x, d\boldsymbol{\omega}) &= \int_{\Omega \setminus E_\epsilon} f(x, d\boldsymbol{\omega}) + \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\boldsymbol{\omega}) + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x, d\boldsymbol{\omega}) \\ &= I_1 + I_2 + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} [f(x, d\boldsymbol{\omega}) - f(x_s, d\boldsymbol{\omega})] + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x_s, d\boldsymbol{\omega}) \\ &= I_1 + I_2 + I_3 + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x_s, d\boldsymbol{\omega}), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\Omega \setminus E_\epsilon} f(x, d\boldsymbol{\omega}), \\ I_2 &= \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\boldsymbol{\omega}), \\ I_3 &= \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} [f(x, d\boldsymbol{\omega}) - f(x_s, d\boldsymbol{\omega})]. \end{aligned}$$

Now, we have,

$$\begin{aligned} I_1 &= \int_{\Omega \setminus E_\epsilon} f(x, d\boldsymbol{\omega}) \leq \int_{\Omega \setminus E_\epsilon} \left[\beta(x) + \sum_{i=1}^r \alpha_i |d\omega_i|^{p_i} + \sum_{i=r+1}^m \eta_i (|d\omega_i|) \right] \\ &\leq \epsilon + \sum_{i=1}^r \int_{\Omega \setminus E_\epsilon} |d\omega_i|^{p_i} + k \text{meas}(\Omega \setminus E_\epsilon), \\ &\leq 2\epsilon + \sum_{i=1}^r \delta_1^i \left(\frac{\epsilon}{k} \right), \end{aligned}$$

where $\delta_1^i(t)$ s are non-negative and increasing functions such that $\delta_1^i(t) \rightarrow 0$ as $t \rightarrow 0$, for each $1 \leq i \leq r$, since $d\omega_i \in L^{p_i}$ for all $1 \leq i \leq r$.

We also have,

$$\begin{aligned}
I_2 &= \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\omega) \\
&\leq \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} \left[\beta(x) + \sum_{i=r+1}^m \eta_i(|d\omega_i|) \right] + \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} \sum_{i=1}^r \alpha_i |d\omega_i|^{p_i} \\
&\leq (M_\epsilon^\beta + k) \text{meas}(E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)) + \sum_{i=1}^r \delta_1^i (\text{meas}(E_\epsilon \setminus (E_\epsilon \cap K_\epsilon))) \\
&\leq (M_\epsilon^\beta + k) \text{meas}(\Omega \setminus K_\epsilon) + \sum_{i=1}^r \delta_1^i (\text{meas}(\Omega \setminus K_\epsilon)) \\
&= (M_\epsilon^\beta + k) (\text{meas}(\Omega \setminus \Omega_\epsilon) + \text{meas}(\Omega_\epsilon \setminus K_\epsilon)) + \sum_{i=1}^r \delta_1^i (\text{meas}(\Omega \setminus \Omega_\epsilon) + \text{meas}(\Omega_\epsilon \setminus K_\epsilon)) \\
&\leq 2(M_\epsilon^\beta + k) \epsilon + \sum_{i=1}^r \delta_1^i (2\epsilon).
\end{aligned}$$

Also, since $x_s \rightarrow x$ as $h \rightarrow \infty$ and f is uniformly continuous on $K_\epsilon \times S_\epsilon$, we have,

$$I_3 = \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} [f(x, d\omega) - f(x_s, d\omega)] \leq \epsilon.$$

Combining, we have,

$$\int_{\Omega} f(x, d\omega) \leq \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x_s, d\omega) + \left[2(M_\epsilon^\beta + k) + 3 \right] \epsilon + \sum_{i=1}^r \left[\delta_1^i \left(\frac{\epsilon}{k} \right) + \delta_1^i (2\epsilon) \right].$$

This implies,

$$\int_{\Omega} f(x, d\omega) \leq \sum_s \int_{D_s} f(x_s, d\omega) + \left[2(M_\epsilon^\beta + k) + 3 \right] \epsilon + \sum_{i=1}^r \left[\delta_1^i \left(\frac{\epsilon}{k} \right) + \delta_1^i (2\epsilon) \right]. \quad (4.32)$$

Estimation of $\int_{\Omega} f(x, d\omega^\nu)$:

We obtain,

$$\begin{aligned}
\int_{\Omega} f(x, d\omega^\nu) &= \int_{\Omega \setminus E_\epsilon} f(x, d\omega^\nu) + \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\omega^\nu) + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x, d\omega^\nu) \\
&= I_1^\nu + I_2^\nu + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} [f(x, d\omega^\nu) - f(x_s, d\omega^\nu)] + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x_s, d\omega^\nu) \\
&= I_1^\nu + I_2^\nu + I_3^\nu + \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} f(x_s, d\omega^\nu) \\
&= I_1^\nu + I_2^\nu + I_3^\nu + I_4^\nu + \sum_s \int_{D_s} f(x_s, d\omega^\nu)
\end{aligned}$$

where

$$\begin{aligned}
I_1^\nu &= \int_{\Omega \setminus E_\epsilon} f(x, d\omega^\nu), \\
I_2^\nu &= \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\omega^\nu), \\
I_3^\nu &= \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} [f(x, d\omega^\nu) - f(x_s, d\omega^\nu)], \\
I_4^\nu &= \sum_s \int_{D_s \setminus (E_\epsilon \cap K_\epsilon \cap D_s)} f(x_s, d\omega^\nu).
\end{aligned}$$

Since for all i such that $p_i = 1$, the sequences $\{d\omega_i^\nu\}$ are equiintegrable, we deduce the existence of the non-negative and increasing functions $\delta_2^i(t)$ s with $\delta_1^i(t) \rightarrow 0$ as $t \rightarrow 0$, for each i such that $p_i = 1$. Proceeding along the same lines as before, we obtain,

$$I_1^\nu = \int_{\Omega \setminus E_\epsilon} f(x, d\omega^\nu) \geq - \int_{\Omega \setminus E_\epsilon} \sum_{\substack{i, \\ p_i=1}} \alpha_i |d\omega_i| \geq - \sum_{\substack{i, \\ p_i=1}} \delta_2^i\left(\frac{\epsilon}{k}\right).$$

Similarly, we have,

$$I_2^\nu = \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} f(x, d\omega^\nu) \geq - \int_{E_\epsilon \setminus (E_\epsilon \cap K_\epsilon)} \sum_{\substack{i, \\ p_i=1}} \alpha_i |d\omega_i| \geq - \sum_{\substack{i, \\ p_i=1}} \delta_2^i(2\epsilon).$$

Also, once again by uniform continuity of f on $K_\epsilon \times S_\epsilon$, we have,

$$I_3^\nu \geq -|I_3^\nu| \geq - \sum_s \int_{E_\epsilon \cap K_\epsilon \cap D_s} |f(x, d\omega^\nu) - f(x_s, d\omega^\nu)| \geq -\epsilon.$$

For the last one, we deduce,

$$\begin{aligned}
I_4^\nu &= \sum_s \int_{D_s \setminus (E_\epsilon \cap K_\epsilon \cap D_s)} f(x_s, d\omega^\nu) \geq - \sum_s \int_{D_s \setminus (E_\epsilon \cap K_\epsilon \cap D_s)} \sum_{\substack{i, \\ p_i=1}} \alpha_i |d\omega_i| \\
&\geq - \sum_{\substack{i, \\ p_i=1}} \delta_2^i(\text{meas}(\Omega \setminus E_\epsilon \cap K_\epsilon)) \\
&\geq - \sum_{\substack{i, \\ p_i=1}} \delta_2^i(\text{meas}(\Omega \setminus E_\epsilon) + \text{meas}(\Omega \setminus K_\epsilon)) \\
&\geq - \sum_{\substack{i, \\ p_i=1}} \delta_2^i(2\epsilon + \frac{\epsilon}{k}).
\end{aligned}$$

Combining, we obtain the estimate,

$$\int_{\Omega} f(x, d\omega^\nu) \geq \sum_s \int_{D_s} f(x_s, d\omega^\nu) - \sum_{\substack{i, \\ p_i=1}} \left[\delta_2^i\left(\frac{\epsilon}{k}\right) + \delta_2^i(2\epsilon) + \delta_2^i(2\epsilon + \frac{\epsilon}{k}) \right] - \epsilon. \quad (4.33)$$

Combining the estimates (4.32) and (4.33) and using Theorem 4.22, we obtain,

$$\begin{aligned}
& \liminf_{\nu \rightarrow \infty} \int_{\Omega} f(x, d\omega^{\nu}) \\
& \geq \liminf_{\nu \rightarrow \infty} \sum_s \int_{D_s} f(x_s, d\omega^{\nu}) - \sum_{\substack{i, \\ p_i=1}} \left[\delta_2^i\left(\frac{\epsilon}{k}\right) + \delta_2^i(2\epsilon) + \delta_2^i\left(2\epsilon + \frac{\epsilon}{k}\right) \right] - \epsilon \\
& \geq \sum_s \int_{D_s} f(x_s, d\omega) - \sum_{\substack{i, \\ p_i=1}} \left[\delta_2^i\left(\frac{\epsilon}{k}\right) + \delta_2^i(2\epsilon) + \delta_2^i\left(2\epsilon + \frac{\epsilon}{k}\right) \right] - \epsilon \\
& \geq \int_{\Omega} f(x, d\omega) - \sum_{\substack{i, \\ p_i=1}} \left[\delta_2^i\left(\frac{\epsilon}{k}\right) + \delta_2^i(2\epsilon) + \delta_2^i\left(2\epsilon + \frac{\epsilon}{k}\right) \right] - \left[2 \left(M_{\epsilon}^{\beta} + k \right) + 4 \right] \epsilon \\
& \qquad \qquad \qquad - \sum_{i=1}^r \left[\delta_1^i\left(\frac{\epsilon}{k}\right) + \delta_1^i(2\epsilon) \right].
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this concludes the proof. ■

Now we are ready to prove the semicontinuity result in full generality.

Theorem 4.25 (Sufficient condition) *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. Let $0 \leq r \leq m$ be an integer. $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i < \infty$ for all $1 \leq i \leq r$ and $p_{r+1} = \dots = p_m = \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \Omega \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function, satisfying the growth condition $(\mathcal{C}_{\mathbf{p}}^x)$ and $\xi \mapsto f(x, \xi)$ is vectorially ext. quasiconvex for a.e $x \in \Omega$. Let $I : W^{d, \mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1}) \rightarrow \mathbb{R}$ defined by*

$$I(\omega) := \int_{\Omega} f(x, d\omega(x)) \, dx, \text{ for all } \omega \in W^{d, \mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1}).$$

Then I is weakly lower semicontinuous in $W^{d, \mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1})$ (weakly $$ in i -th factor if $p_i = \infty$).*

Proof We just need to show that we can reduce the theorem to the particular case proved in Theorem 4.24. We divide the proof into several steps.

Step 1 We begin by showing that we can assume f satisfies the following growth condition,

$$- \sum_{\substack{i \\ p_i=1}} \alpha_i |\xi_i| \leq f(x, \xi) \leq \beta(x) + \sum_{\substack{i \\ p_i=1}} \alpha_i |\xi_i| + \sum_{\substack{i \\ 1 < p_i < \infty}} g_i(x) |\xi_i| + \sum_{i=r+1}^m \eta_i(|\xi_i|), \quad (\mathcal{C}_{\mathbf{p}}^{x''})$$

We choose a sequence

$$\omega^{\nu} \rightharpoonup \omega \quad \text{in } W^{d, \mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1}).$$

Since

$$\omega_i^{\nu} \xrightarrow{*} \omega_i \quad \text{in } W^{d, \infty}(\Omega; \Lambda^{k_i-1}) \quad \text{for every } r+1 \leq i \leq m,$$

we have,

$$\|d\omega_i^{\nu}\|_{L^{\infty}} \leq \gamma_i \quad \text{for every } r+1 \leq i \leq m,$$

for some constants $\gamma_i > 0$.

Also, if $1 \leq q_i < p_i$, then for every $\varepsilon > 0$, there exists a constant $k_i = k_i(\varepsilon) > 0$ such that

$$\varepsilon |\xi_i|^{p_i} + k_i \leq \alpha_i |\xi_i|^{q_i} \quad \text{for all } \xi_i \in \Lambda^{k_i}.$$

We set

$$k := \sum_{\substack{i \\ 1 < p_i < \infty}} k_i + \sum_{i=r+1}^m \eta_i(\gamma_i).$$

Now we define,

$$f_\varepsilon(x, \xi) = f(x, \xi) + \beta(x) + \varepsilon \sum_{\substack{i \\ 1 < p_i < \infty}} |\xi|^{p_i} + k.$$

Since f satisfies (\mathcal{C}_p^x) , f_ε satisfies $\mathcal{C}_p^{x'}$ for every $\varepsilon > 0$. Also it is clear that f_ε is also a Carathéodory function and $\xi \mapsto f_\varepsilon(x, \xi)$ is vectorially ext. quasiconvex for a.e $x \in \Omega$, for every $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$, the semicontinuity result for f follows from the semicontinuity results for f_ε . Hence, we can assume f satisfies $\mathcal{C}_p^{x''}$.

Step 2 Now we show that we can assume that f satisfies $\mathcal{C}_p^{x'}$. Of course, the only thing to show is that it is possible to replace the functions $g_i(x)$ with constants. We define, for every natural number μ ,

$$\phi^\mu(x) := \begin{cases} 1 & \text{if } \max_{\substack{i \\ 1 < p_i < \infty}} g_i(x) \leq \mu \\ \frac{\mu}{\max_{\substack{i \\ 1 < p_i < \infty}} g_i(x)} & \text{if otherwise.} \end{cases}$$

Defining

$$f_\mu(x, \xi) := \phi^\mu(x) f(x, \xi),$$

we see immediately that f_μ is a Carathéodory function satisfying $\mathcal{C}_p^{x'}$ and $\xi \mapsto f_\mu(x, \xi)$ is vectorially ext. quasiconvex for a.e $x \in \Omega$. Furthermore,

$$f(x, \xi) = \sup_{\mu} f_\mu(x, \xi) = \lim_{\mu \rightarrow \infty} f_\mu(x, \xi).$$

Thus, the theorem for f_μ implies,

$$\begin{aligned} \liminf_{\nu \rightarrow \infty} \int_{\Omega} f(x, d\omega^\nu) &\geq \liminf_{\nu \rightarrow \infty} \int_{\Omega} f_\mu(x, d\omega^\nu) \\ &\geq \int_{\Omega} f_\mu(x, d\omega). \end{aligned}$$

Taking the supremum over μ on the right hand side proves the result. This shows that we can assume that f satisfies $\mathcal{C}_p^{x'}$.

Step 3 Now we show that we can assume Ω is an open cube with sides parallel to the axes. Since we can treat each cube separately, it is enough to show that Ω can be taken to be a finite

union of disjoint such cubes. To this end, we choose $\Omega_\mu \subset \Omega$ to be a finite union of disjoint open cubes, with sides parallel to the axes, of side length μ . Since for all i such that $p_i = 1$, we have,

$$d\omega_i^\nu \rightharpoonup d\omega_i \quad \text{in } L^1(\Omega, \Lambda^{k_i}),$$

the sequences $\{|d\omega_i^\nu|\}$ are equiintegrable. Hence, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\text{meas } A \leq \delta \quad \Rightarrow \quad \int_A \sum_{\substack{i \\ p_i=1}} \alpha_i |d\omega_i^\nu| \leq \epsilon.$$

Choosing μ large enough, we can ensure

$$\text{meas}(\Omega \setminus \Omega_\mu) \leq \delta.$$

Thus, we obtain, using $\mathcal{C}_p^{x'}$,

$$\begin{aligned} \int_{\Omega} f(x, d\omega^\nu) &\geq \int_{\Omega_\mu} f(x, d\omega^\nu) + \int_{\Omega \setminus \Omega_\mu} f(x, d\omega^\nu) \\ &\geq \int_{\Omega_\mu} f(x, d\omega^\nu) - \int_{\Omega \setminus \Omega_\mu} \sum_{\substack{i \\ p_i=1}} \alpha_i |d\omega_i^\nu| \\ &\geq \int_{\Omega_\mu} f(x, d\omega^\nu) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we deduce the semicontinuity result for Ω from the ones for Ω_μ by letting $\mu \rightarrow \infty$.

Step 4 All that remains to show is that we can restrict ourselves to sequences with the additional property of $\{|d\omega_i^\nu|^{p_i}\}$ being equiintegrable for each i such that $1 < p_i < \infty$. This is done in a similar manner as in Step 1 of the proof of Theorem 4.22 above. This concludes the reduction of the theorem to Theorem 4.24 and finishes the proof. ■

Failure of semicontinuity in $W^{d,p}$ for general functional

Vectorial ext. quasiconvexity of the map $\xi \mapsto f(x, \omega, \xi)$, along with usual growth conditions, is not sufficient for weak lower semicontinuity in $W^{d,p}$ of functionals with explicit dependence on ω , i.e for functionals of the form,

$$\int_{\Omega} f(x, \omega, d\omega) \, dx.$$

For example, even when $m = 1$, for $n \geq 3, k \geq 2$, theorem 3.62 gives a counter-example. However, if $k_i = 1$ for all $1 \leq i \leq m$, the functional $\int_{\Omega} f(x, \omega, d\omega) \, dx$ is weakly lower semicontinuous in $W^{d,p}$, precisely because in this case $W^{d,p}$ and $W^{1,p}$ are the same space. Indeed, it is possible to show the more general result that the functional $\int_{\Omega} f(x, \omega, d\omega(x)) \, dx$ is always weakly lower semicontinuous in $W^{1,p}$ with appropriate growth conditions on f .

Semicontinuity in $W^{1,p}$ for general functional

We first define the appropriate growth conditions in this setting.

Definition 4.26 (Growth condition) Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $f : \Omega \times \Lambda^{\mathbf{k}-1} \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function.

f is said to be of growth $(\mathcal{C}_{\mathbf{p}}^{x,u})$, if, for almost every $x \in \Omega$ and for every $(\mathbf{u}, \boldsymbol{\xi}) \in \Lambda^{\mathbf{k}-1} \times \Lambda^{\mathbf{k}}$, f satisfies,

$$-\beta(x) - \sum_{i=1}^m G_i^l(u_i, \xi_i) \leq f(x, \mathbf{u}, \boldsymbol{\xi}) \leq \beta(x) + \sum_{i=1}^m G_i^u(u_i, \xi_i), \quad (\mathcal{C}_{\mathbf{p}}^{x,u})$$

where $\beta \in L^1(\Omega)$ is nonnegative and the functions G_i^l s in the lower bound and the functions G_i^u s in the upper bound has the following form:

- If $p_i = 1$, then,

$$G_i^l(u_i, \xi_i) = G_i^u(u_i, \xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$G_i^l(u_i, \xi_i) = \alpha_i (|\xi_i|^{q_i} + |u_i|^{r_i})$$

and

$$G_i^u(u_i, \xi_i) = g_i(x, u_i) |\xi_i|^{p_i},$$

for some $1 \leq q_i < p_i$, $1 \leq r_i < np_i/(n - p_i)$ if $p_i < n$ and $1 \leq r_i < \infty$ if $p_i \geq n$, g_i is a nonnegative Carathéodory function and for some constant $\alpha_i \geq 0$.

- If $p_i = \infty$, then,

$$G_i^l(u_i, \xi_i) = G_i^u(u_i, \xi_i) = \eta_i (|u_i|, |\xi_i|).$$

for some nonnegative, continuous, increasing (in each argument) function η_i .

With these growth conditions on f , we can show that the functional $\int_{\Omega} f(x, \boldsymbol{\omega}, d\boldsymbol{\omega}(x)) dx$ is always weakly lower semicontinuous in $W^{1,\mathbf{p}}$.

The proof is very similar to the proof of Theorem 4.25. In this case too, it is possible to derive all the necessary estimates after freezing both x and $\boldsymbol{\omega}$. Some modifications are required to handle the explicit dependence on $\boldsymbol{\omega}$, but these modifications essentially use the Sobolev embedding and is quite standard (see theorem 8.8 and theorem 8.11 in [25] for the classical case). We state the theorem below and omit the proof.

Theorem 4.27 Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \Omega \times \Lambda^{\mathbf{k}-1} \times \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ be a Carathéodory function, satisfying the growth condition $(\mathcal{C}_{\mathbf{p}}^{x,u})$ and $\boldsymbol{\xi} \mapsto f(x, \mathbf{u}, \boldsymbol{\xi})$ is vectorially ext. quasiconvex for a.e $x \in \Omega$ and for every $\mathbf{u} \in \Lambda^{\mathbf{k}-1}$. Let $I : W^{1,\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1}) \rightarrow \mathbb{R}$ defined by

$$I(\boldsymbol{\omega}) := \int_{\Omega} f(x, \boldsymbol{\omega}, d\boldsymbol{\omega}) dx, \text{ for all } \boldsymbol{\omega} \in W^{1,\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1}).$$

Then I is weakly lower semicontinuous in $W^{1,\mathbf{p}}(\Omega; \mathbf{\Lambda}^{k-1})$ (weakly $*$ in i -th factor if $p_i = \infty$).

Corollary 4.28 Let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth. Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ be a Carathéodory function and satisfies, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{m \times n}$, the growth condition,

$$-\beta(x) - \sum_{i=1}^m G_i^l(u_i, \xi_i) \leq f(x, u, \xi) \leq \beta(x) + \sum_{i=1}^m G_i^u(u_i, \xi_i),$$

where u_i is the i -th component of $u = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $\xi_i = (\xi_{i1}, \dots, \xi_{in})$ is the i -th row

of the matrix $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \in \mathbb{R}^{m \times n}$, $\beta \in L^1(\Omega)$ is nonnegative and the functions G_i^l s in the lower bound and the functions G_i^u s in the upper bound has the following form:

- If $p_i = 1$, then,

$$G_i^l(u_i, \xi_i) = G_i^u(u_i, \xi_i) = \alpha_i |\xi_i| \quad \text{for some constant } \alpha_i \geq 0.$$

- If $1 < p_i < \infty$, then,

$$G_i^l(u_i, \xi_i) = \alpha_i (|\xi_i|^{q_i} + |u_i|^{r_i})$$

and

$$G_i^u(u_i, \xi_i) = g_i(x, u_i) |\xi_i|^{p_i},$$

for some $1 \leq q_i < p_i$, $1 \leq r_i < np_i/(n - p_i)$ if $p_i < n$ and $1 \leq r_i < \infty$ if $p_i \geq n$, g_i is a nonnegative Carathéodory function and for some constant $\alpha_i \geq 0$.

- If $p_i = \infty$, then,

$$G_i^l(u_i, \xi_i) = G_i^u(u_i, \xi_i) = \eta_i(|u_i|, |\xi_i|).$$

for some nonnegative, continuous, increasing (in each argument) function η_i .

Also let $\xi \mapsto f(x, u, \xi)$ is quasiconvex for a.e $x \in \Omega$ and for every $u \in \mathbb{R}^m$. Let $\{u^\nu\}$ be a sequence such that for every $1 \leq i \leq m$, we have,

$$u_i^\nu \rightharpoonup u_i \quad \text{in } W^{1,p_i} \quad (\overset{*}{\rightharpoonup} \text{ if } p_i = \infty),$$

for some $u_i \in W^{1,p_i}(\Omega)$, then,

$$\liminf_{\nu \rightarrow \infty} \int_{\Omega} f(x, u^\nu, \nabla u^\nu) dx \geq \int_{\Omega} f(x, u, \nabla u) dx.$$

Remark 4.29 The improvement from the classical results is that the p_i s are allowed to be different from one another. If we take, $p_i = p$ for every $1 \leq i \leq m$, then we obtain the classical results.

4.5 Weak continuity

The aim of this section is to characterize all weakly continuous functions. We shall show that the wedge products play the same role in this setting that determinants and adjugates play in classical calculus of variations. Here we shall restrict our analysis to classical wedge products, i.e. when wedge products make sense as differential forms with L^1 components. However, distributional Jacobian and distributional adjugates are well studied in classical calculus of variations, not only in usual setting, but even in fractional Sobolev spaces for mapping taking values in manifolds (see for example Brezis-Bourgain-Mironescu[15], Brezis-Bourgain-Mironescu[16], Brezis-Nguyen[17]). Also, weak wedge products has also been introduced and studied, most notable in connection to geometric function theory and quasiconformal mappings (see Iwaniec[38]). Let us begin with the definition.

Definition 4.30 (Weak continuity) Let $k = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Lambda^k \rightarrow \mathbb{R}$ be continuous. We say that f is weakly continuous on $W^{d,p}(\Omega; \Lambda^{k-1})$, if for every sequence $\{\omega^\nu\}_{\nu=1}^\infty = \{(\omega_1^\nu, \dots, \omega_m^\nu)\}_{\nu=1}^\infty \subset W^{d,p}(\Omega; \Lambda^{k-1})$ and every $\omega = (\omega_1, \dots, \omega_m) \in W^{d,p}(\Omega; \Lambda^{k-1})$ satisfying

$$\omega^\nu \rightharpoonup \omega \text{ in } W^{d,p}(\Omega; \Lambda^{k-1}) \quad (* \text{ if } p_i = \infty),$$

we have

$$f(d\omega^\nu) \rightharpoonup f(d\omega) \text{ in } \mathcal{D}'(\Omega).$$

4.5.1 Necessary condition

Theorem 4.31 (Necessary condition) Let $k = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $f : \Lambda^k \rightarrow \mathbb{R}$ be weakly continuous on $W^{d,\infty}(\Omega; \Lambda^k)$. Then, f is vectorially ext. one affine, and hence, is of the form

$$f(\xi) = \sum_{\substack{\alpha, \\ 0 \leq |k\alpha| \leq n}} \langle c_\alpha; \xi^\alpha \rangle \text{ for all } \xi \in \Lambda^k, \quad (4.34)$$

where $c_\alpha \in \Lambda^{|k\alpha|}(\mathbb{R}^n)$, for every $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |k\alpha| \leq n$.

Remark 4.32 Since f being weakly continuous in $W^{d,\infty}(\Omega; \Lambda^{k-1})$ is a necessary condition for f to be weakly continuous in $W^{d,p}(\Omega; \Lambda^{k-1})$ for any $p = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$, f being vectorially ext. one affine is a necessary condition for weak continuity in $W^{d,p}(\Omega; \Lambda^{k-1})$ as well.

Proof Since f is weakly continuous on $W^{d,\infty}(\Omega; \Lambda^k)$, then for any $\phi \in C_c^\infty(\Omega)$ and for any sequence $\{\omega^\nu\}_{\nu=1}^\infty \subset W^{d,p}(\Omega; \Lambda^{k-1})$ with

$$\omega^\nu \rightharpoonup \omega \text{ in } W^{d,p}(\Omega; \Lambda^{k-1}) \quad (* \text{ if } p_i = \infty),$$

we have,

$$\liminf_{\nu \rightarrow \infty} \int_{\Omega} \phi(x) f(d\omega^\nu) = \int_{\Omega} \phi(x) f(d\omega) \text{ meas}(\Omega).$$

Thus, defining

$$I(x, \omega) := \int_{\Omega} \phi(x) f(d\omega) \quad \text{for any } \omega \in W^{d, \infty}(\Omega; \Lambda^k)$$

and using Theorem 4.14, we obtain that

$$\xi \mapsto \phi(x) f(\xi)$$

must be vectorially ext. quasilinear. Since $\phi \in C_c^\infty(\Omega)$ is arbitrary, this implies $\xi \mapsto f(\xi)$ must be vectorially ext. quasilinear. This finishes the proof. ■

4.5.2 Sufficient condition

Now we shall present the results about sufficient conditions for weak continuity. First, we state a theorem which was proved in Robin-Rogers-Temple [57], using Hodge decomposition.

Theorem 4.33 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\alpha = (\alpha_1, \dots, \alpha_m)$ where $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |\mathbf{k}\alpha| \leq n$ and let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 < p_i < \infty$ for all $1 \leq i \leq m$.*

Suppose $1 = \sum_{i=1}^m \frac{\alpha_i}{p_i}$ and $\mathbf{v}_\nu \rightharpoonup \mathbf{v}$ in $L^{\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}})$ with

$$d\mathbf{v}^\nu \in \text{a compact set of } W^{-1, \mathbf{p}}(\Omega; \Lambda^{\mathbf{k}+1}).$$

Then

$$\mathbf{v}_\nu^\alpha \rightharpoonup \mathbf{v}^\alpha \quad \text{in } \mathcal{D}'(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)).$$

Theorem 4.34 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded. Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, $\alpha = (\alpha_1, \dots, \alpha_m)$ where $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |\mathbf{k}\alpha| \leq n$ and let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 < p_i \leq \infty$ for all $1 \leq i \leq m$.*

Suppose $1 \geq \frac{1}{q} = \sum_{i=1}^m \frac{\alpha_i}{p_i}$ and $d\omega_\nu \rightharpoonup d\omega$ in $L^{\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}})$.

Then the following holds true.

(i) *If $q > 1$, then*

$$d\omega_\nu^\alpha \rightharpoonup d\omega^\alpha \quad \text{in } L^q(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)) \quad (\overset{*}{\rightharpoonup} \text{ if } q = \infty).$$

(ii) *if $q = 1$, but $1 < p_i < \infty$ for all $1 \leq i \leq m$, then*

$$d\omega_\nu^\alpha \rightharpoonup d\omega^\alpha \quad \text{in } \mathcal{D}'(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)).$$

Remark 4.35 When $q = 1$, weak convergence in L^1 does not hold, in general, even when $1 < p_i < \infty$ for all $1 \leq i \leq m$. There exist sequences $\omega_\nu \rightharpoonup \omega$ in $W^{d,p}(\Omega; \Lambda^{k-1})$ with $q = 1$ such that $\omega_\nu^\alpha \not\rightharpoonup \omega^\alpha$ in $L^1(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n))$. Even when $k_i = 1$ and $p_i = m$ for all $1 \leq i \leq m$, such counter-examples are known in classical vectorial calculus of variations.

Proof Since $d \circ d = 0$, second conclusion follows directly from Theorem 4.33. So we only prove the first conclusion.

Step 1 We first prove

$$d\omega_\nu^\alpha \rightharpoonup d\omega^\alpha \quad \text{in } \mathcal{D}'(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)).$$

If $p_i = \infty$ for some $1 \leq i \leq m$, we can suppose, by renaming the variables if necessary, that $1 < p_i < \infty$ for all $1 \leq i < r$ and $p_r = \dots = p_m = \infty$, for some $1 \leq r \leq m$. Now since $1 > \frac{1}{q}$, for every $r \leq i \leq m$, there exist numbers $1 < \tilde{p}_i < \infty$, such that,

$$1 > \sum_{i=1}^{r-1} \frac{\alpha_i}{p_i} + \sum_{i=r}^m \frac{\alpha_i}{\tilde{p}_i}.$$

Since weak convergence in L^∞ implies weak convergence in $L^{\tilde{p}_i}$, this means that we can always assume that $1 < p_i < \infty$ for all $1 \leq i \leq m$, without loss of generality.

Now we can also choose numbers q_i s such that $1 < q_i < p_i < \infty$ for all $1 \leq i \leq m$ and

$$1 = \sum_{i=1}^m \frac{\alpha_i}{q_i}.$$

Now let $\phi \in C_c^\infty(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n))$. Define $f : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ as

$$f(x, \xi) = \langle \phi(x), \xi^\alpha \rangle.$$

Now we have, by Young's inequality,

$$|\xi^\alpha| \leq \prod_{i=1}^m |\xi_i|^{\alpha_i} \leq \sum_{i=1}^m \frac{\alpha_i}{q_i} |\xi_i|^{q_i}.$$

Since $q_i < p_i$ for all $1 \leq i \leq m$, this implies that both f and $-f$ satisfies the growth condition (\mathcal{C}_p^x) . Also, $\xi \mapsto f(x, \xi)$ and $\xi \mapsto -f(x, \xi)$ are both vectorially ext. quasiconvex for a.e $x \in \Omega$. Hence, applying Theorem 4.25 to both f and $-f$, we deduce,

$$\lim_{\nu \rightarrow \infty} \int_{\Omega} \langle \phi(x), d\omega_\nu^\alpha \rangle = \int_{\Omega} \langle \phi(x), d\omega^\alpha \rangle.$$

Since $\phi \in C_c^\infty(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n))$, this proves the convergence in the sense of distributions.

Step 2 The hypotheses imply easily that $\{d\omega_\nu^\alpha\}$ is uniformly bounded in $L^q(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n))$. Since $q > 1$, this implies,

$$d\omega_\nu^\alpha \rightharpoonup \zeta \quad \text{in } L^q(\Omega; \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)) \quad (\xrightarrow{*} \text{ if } q = \infty).$$

But by the convergence in distributions and uniqueness of the weak limit, we must have,

$$\zeta = d\omega^\alpha.$$

This finishes the proof. ■

Theorem 4.36 (Sufficient condition) *Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $f : \Lambda^{\mathbf{k}} \rightarrow \mathbb{R}$ is vectorially ext. one affine, and hence, is of the form*

$$f(\xi) = \sum_{\substack{\alpha, \\ 0 \leq |\mathbf{k}\alpha| \leq n}} \langle c_\alpha; \xi^\alpha \rangle \text{ for all } \xi \in \Lambda^{\mathbf{k}},$$

where $c_\alpha \in \Lambda^{|\mathbf{k}\alpha|}(\mathbb{R}^n)$, for every $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$ and $0 \leq |\mathbf{k}\alpha| \leq n$.

Let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 < p_i \leq \infty$ for all $1 \leq i \leq m$, be such that,

$$\max \left\{ \sum_{i=1}^m \frac{\alpha_i}{p_i} : c_\alpha \neq 0 \right\} := \frac{1}{q} \leq 1.$$

Then for any sequence $\{\omega_\nu\} \subset W^{d,\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1})$ such that

$$d\omega_\nu \rightharpoonup d\omega \text{ in } L^{\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}}),$$

for some $\omega \in W^{d,\mathbf{p}}(\Omega; \Lambda^{\mathbf{k}-1})$, we have,

(i) If $q > 1$, then

$$f(d\omega_\nu) \rightharpoonup f(d\omega) \text{ in } L^q(\Omega) \quad (\xrightarrow{*} \text{ if } q = \infty).$$

(ii) if $q = 1$, but $1 < p_i < \infty$ for all $1 \leq i \leq m$, then

$$f(d\omega_\nu) \rightharpoonup f(d\omega) \text{ in } \mathcal{D}'(\Omega).$$

Proof This is an immediate corollary of Theorem 4.34. ■

Chapter 5

Other types of functionals

5.1 Introduction

In this chapter we are going to explore the scope of carrying out our program for more general functionals. We work with functionals depending on both d and δ of a single unknown differential form, i.e functionals of the form,

$$\int_{\Omega} f(d\omega, \delta\omega).$$

We shall define the correct convexity notions, establish the characterization theorem for the corresponding affine functions and present a few simple properties and finally, existence theorems for a few minimization problems. In contrast to the case with the last two chapters, we shall show here that these functionals do not suffer from the lack of coercivity by virtue of Gaffney inequality and hence existence can be obtained as soon as convexity conditions ensure weak lower semicontinuity. Also, as we shall show in the characterization theorem (Theorem 5.11), non-linear ‘quasiaffine’ functions in these case can be nonlinear either with respect to $d\omega$ or $\delta\omega$, but *not with respect to both of them*. This makes the situation considerably barren. Though one would naturally anticipate a richer situation with respect to convexity than the case of ext. convexity or int. convexity notions, this fact strips away much of that possibility. As we shall show, at least at the level of affinity, ext-int. affinity notions are essentially the same as ext. affinity or int. affinity notions.

Hence in terms of coercivity, the situation is considerably more simpler than the last two chapters and in terms of convexity notions, not really any more complicated in any essential way. Faced with these results, we see little reason to attempt to carry out our program in full. So in this chapter, we are not really going to spend a lot of energy on these, but just prove the basic results we already mentioned. However, the existence theorems can be useful in some applications.

Further generalizations are also possible. Generalizations to functions which depend upon exterior derivatives of some forms and codifferential of some forms and both exterior derivative and codifferential of some forms, e.g. convexity notions to treat functionals of the form

$$\int_{\Omega} f(d\omega_1, \delta\omega_2, d\omega_3, \delta\omega_3),$$

can be defined easily, though at this point it is not clear if there is any new insight to be gained from treating such generalities. At this point, such generalizations seem to be rather routine exercises towards somewhat contrived and artificial generalizations.

5.2 Notions of Convexity

5.2.1 Definitions

We start with the different notions of convexity and affinity.

Definition 5.1 Let $1 \leq k \leq n - 1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(i) We say that f is *ext-int. one convex*, if the function

$$g : t \rightarrow g(t) = f(\xi + t\alpha \wedge \beta, \eta + t\alpha \lrcorner \beta)$$

is convex for every $\xi \in \Lambda^{k+1}$, $\eta \in \Lambda^{k-1}$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^k$. If the function g is affine we say that f is *ext-int. one affine*.

(ii) A Borel measurable and locally bounded function f is said to be *ext-int. quasiconvex*, if

$$\int_{\Omega} f(\xi + d\omega, \eta + \delta\omega) \geq f(\xi, \eta) \text{ meas } \Omega$$

for every bounded open set Ω , for every $\xi \in \Lambda^{k+1}$, $\eta \in \Lambda^{k-1}$ and for every $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$. If equality holds, we say that f is *ext-int. quas affine*.

(iii) We say that f is *ext-int. polyconvex*, if there exists a convex function

$$F : \Lambda^{k+1} \times \Lambda^{2(k+1)} \times \dots \times \Lambda^{\left[\frac{n}{k+1}\right](k+1)} \times \Lambda^{n-k+1} \times \Lambda^{2(n-k+1)} \times \dots \times \Lambda^{\left[\frac{n}{n-k+1}\right](n-k+1)} \rightarrow \mathbb{R}$$

such that

$$f(\xi, \eta) = F\left(\xi, \xi^2, \dots, \xi^{\left[\frac{n}{k+1}\right]}, *\eta, (*\eta)^2, \dots, (*\eta)^{\left[\frac{n}{n-k+1}\right]}\right).$$

If F is affine, we say that f is *ext-int. polyaffine*.

We close this subsection with another notion of convexity, which will not be used much in the sequel, but is, however, interesting. Unlike the notions discussed in the third chapter, in this case there is the possibility of another related sets of notions of convexity. The classical notion of a separately convex function (see [25]) is easy. The function is required to be convex in each variable separately. Since convexity is exactly the same as the classical convexity in both single derivative and both derivative case of functions of differential forms, the notion of separately convex functions are also the same in all these cases. But unlike the classical case or the case of a single differential form with single derivative, it is now possible to talk of separately ext-int. polyconvex, separately ext-int. quasiconvex and separately ext-int. one convex functions. Though we will not be exploring these questions much further, for the sake of completeness we define them below:

Definition 5.2 Let $1 \leq k \leq n - 1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$.

(a) We say that f is separately ext-int. one convex, if ,

(i) The function $g(\xi) = f(\xi, \eta)$ is ext. one convex for every $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$.

(ii) The function $h(\eta) = f(\xi, \eta)$ is int. one convex for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$.

(b) A Borel measurable and locally bounded function f is said to be separately ext-int. quasiconvex, if

(i) The function $g(\xi) = f(\xi, \eta)$ is ext. quasiconvex for every $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$.

(ii) The function $h(\eta) = f(\xi, \eta)$ is int. quasiconvex for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$.

(c) We say that f is separately ext-int. polyconvex, if,

(i) The function $g(\xi) = f(\xi, \eta)$ is ext. polyconvex for every $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$.

(ii) The function $h(\eta) = f(\xi, \eta)$ is int. polyconvex for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$.

5.2.2 Main properties

Several important properties can be deduced for these functions in an analogous way as was done in chapter 3. However, we only prove the ones we shall use. We start with the most basic one, the general relationship between the different notions of convexity. They are related as follows.

Theorem 5.3 Let $1 \leq k \leq n-1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$. Then

$$f \text{ convex} \Rightarrow f \text{ ext-int. polyconvex} \Rightarrow f \text{ ext-int. quasiconvex} \Rightarrow f \text{ ext-int. one convex.}$$

Moreover if $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ is ext. one convex, then f is locally Lipschitz. If, in addition f is C^2 , then for every $\xi \in \Lambda^{k+1}, \eta \in \Lambda^{k-1}, \alpha \in \Lambda^1$ and $\beta \in \Lambda^k$,

$$\begin{aligned} \sum_{I, J \in \mathcal{T}^{k+1}} \frac{\partial^2 f(\xi, \eta)}{\partial \xi_I \partial \xi_J} (\alpha \wedge \beta)_I (\alpha \wedge \beta)_J + \sum_{I, J \in \mathcal{T}^{k-1}} \frac{\partial^2 f(\xi, \eta)}{\partial \eta_I \partial \eta_J} (\alpha \lrcorner \beta)_I (\alpha \lrcorner \beta)_J \\ + \sum_{\substack{I \in \mathcal{T}^{k+1} \\ J \in \mathcal{T}^{k-1}}} \frac{\partial^2 f(\xi, \eta)}{\partial \xi_I \partial \eta_J} (\alpha \wedge \beta)_I (\alpha \lrcorner \beta)_J \geq 0. \end{aligned}$$

Proof (i) The first implication, i.e f convex $\Rightarrow f$ ext-int. polyconvex is trivial. The second implication, i.e f ext-int. polyconvex $\Rightarrow f$ ext-int. quasiconvex follows from Jensen inequality in the same way as in the proof of Theorem 3.37. Also, the last implication, i.e f ext-int. quasiconvex $\Rightarrow f$ ext-int. one convex. can be proved directly in an analogous way. However, using Theorem 5.17, the result follows from results in classical calculus of variations (see Theorem 5.3 in [25]).

(ii) The fact that f is locally Lipschitz follows from the observation that any ext-int. one convex function is in fact separately convex. Such functions are known to be locally Lipschitz (cf. Theorem 2.31 in [25]).

(iii) We next assume that f is C^2 . By definition the function

$$g : t \rightarrow g(t) = f(\xi + t\alpha \wedge \beta, \eta + t\alpha \lrcorner \beta)$$

is convex for every $\xi \in \Lambda^{k+1}, \eta \in \Lambda^{k-1}, \alpha \in \Lambda^1$ and $\beta \in \Lambda^k$. Since f is C^2 , we get the claim from the fact that $g''(0) \geq 0$. ■

Theorem 5.4 *Let $1 \leq k \leq n-1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$. Then*

(i) *If k and n are both even integer or $n = 2k-1, 2k$ or $2k+1$, then*

$$f \text{ convex} \Leftrightarrow f \text{ ext-int. polyconvex}.$$

Proof If both n and k are even, then since $k+1$ and $n-k+1$ are both odd integers, all the terms ξ^s and $(*\eta)^s$ in the definition of ext-int. polyconvexity are 0 for $s \geq 2$. If $n = 2k-1, 2k$ or $2k+1$, then also both $\left[\frac{n}{k+1}\right]$ and $\left[\frac{n}{n-k+1}\right]$ is equal to 1, implying the result. ■

5.3 The quasilinear case

Now we move on to proving the characterization theorem for ext-int. quasilinear functions.

5.3.1 Some preliminary results

We begin with a few lemmas.

Lemma 5.5 *Let $1 \leq k \leq n-1$ and let $a \in \Lambda^1(\mathbb{R}^n), b \in \Lambda^k(\mathbb{R}^n)$ be given. Then there exists $c \in \Lambda^1(\mathbb{R}^n), d \in \Lambda^k(\mathbb{R}^n)$ such that $c \wedge d = a \wedge b$ and $c \lrcorner d = 0$.*

Proof We choose $c = \frac{1}{|a|}a$ and $d = \frac{1}{|a|}a \lrcorner (a \wedge b)$. Then clearly,

$$c \lrcorner d = \frac{1}{|a|^2}a \lrcorner (a \lrcorner (a \wedge b)) = 0.$$

Also,

$$c \wedge d = \frac{1}{|a|^2}a \wedge (a \lrcorner (a \wedge b)) = \frac{1}{|a|^2} \{ |a|^2(a \wedge b) - a \lrcorner (a \wedge (a \wedge b)) \} = a \wedge b.$$

■

Similarly, we have,

Lemma 5.6 *Let $1 \leq k \leq n-1$ and let $a \in \Lambda^1(\mathbb{R}^n), b \in \Lambda^k(\mathbb{R}^n)$ be given. Then there exists $c \in \Lambda^1(\mathbb{R}^n), d \in \Lambda^k(\mathbb{R}^n)$ such that $c \lrcorner d = a \lrcorner b$ and $c \wedge d = 0$.*

Proof We choose $c = \frac{1}{|a|}a$ and $d = \frac{1}{|a|}a \wedge (a \lrcorner b)$. Then clearly,

$$c \wedge d = \frac{1}{|a|^2}a \wedge (a \wedge (a \lrcorner b)) = 0.$$

Also,

$$c \lrcorner d = \frac{1}{|a|^2}a \lrcorner (a \wedge (a \lrcorner b)) = \frac{1}{|a|^2} \{ |a|^2(a \lrcorner b) - a \wedge (a \lrcorner (a \lrcorner b)) \} = a \lrcorner b.$$

■

Lemma 5.7 *Let $1 \leq k \leq n - 1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext-int. one convex. Then the following holds true:*

- (i) *The function $g(\xi) = f(\xi, \eta)$ is ext. one convex for every $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$.*
- (ii) *The function $h(\eta) = f(\xi, \eta)$ is int. one convex for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$.*

Proof (i) We need to show that for any η , the function $G(t) = g(\xi + ta \wedge b)$ is convex in t for all $a \in \Lambda^1(\mathbb{R}^n)$, $b \in \Lambda^k(\mathbb{R}^n)$. Now by lemma 5.5, for given $a \in \Lambda^1(\mathbb{R}^n)$, $b \in \Lambda^k(\mathbb{R}^n)$, we can find $c \in \Lambda^1(\mathbb{R}^n)$, $d \in \Lambda^k(\mathbb{R}^n)$ such that $c \wedge d = a \wedge b$ and $c \lrcorner d = 0$. Hence,

$$G(t) = g(\xi + ta \wedge b) = f(\xi + ta \wedge b, \eta) = f(\xi + tc \wedge d, \eta + c \lrcorner d),$$

which is convex in t , since f is ext-int. one convex. This establishes the claim.

(ii) We need to show that for any ξ , the function $H(t) = h(\eta + ta \lrcorner b)$ is convex in t for all $a \in \Lambda^1(\mathbb{R}^n)$, $b \in \Lambda^k(\mathbb{R}^n)$. Now by lemma 5.6, for given $a \in \Lambda^1(\mathbb{R}^n)$, $b \in \Lambda^k(\mathbb{R}^n)$, we can find $c \in \Lambda^1(\mathbb{R}^n)$, $d \in \Lambda^k(\mathbb{R}^n)$ such that $c \lrcorner d = a \lrcorner b$ and $c \wedge d = 0$. Hence,

$$H(t) = h(\eta + ta \lrcorner b) = f(\xi, \eta + ta \lrcorner b) = f(\xi + tc \wedge d, \eta + c \lrcorner d),$$

which is convex in t , since f is ext-int. one convex. This establishes the claim. ■

Remark 5.8 *The converse of this lemma fails miserably. An easy counter-example is provided by the function $f(\xi, \eta) = \eta \langle \xi, e^1 \wedge e^2 \rangle$ in the case $k = 1, n = 2$. In this case, where $g(\xi)$ and $h(\eta)$ mentioned above are clearly affine, but $f(ta \wedge b, ta \lrcorner b) = t^2(a_1b_2 - a_2b_1)(a_1b_1 + a_2b_2)$. Hence by choosing $b_1 = 0$ and $a_1 = -a_2$, we get $f(ta \wedge b, ta \lrcorner b) = -(a_2b_2)^2t^2$, which is concave in t . This counter-example can be generalized much further.*

Now we present a corollary which is an immediate consequence of theorem 5.7.

Corollary 5.9 *Let $1 \leq k \leq n - 1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext-int. one affine. Then the following holds true:*

- (i) *The function $g(\xi) = f(\xi, \eta)$ is ext. one affine for every $\eta \in \Lambda^{k-1}(\mathbb{R}^n)$.*
- (ii) *The function $h(\eta) = f(\xi, \eta)$ is int. one affine for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$.*

Remark 5.10 *The converse of this is the subject matter of the main theorem of this section, presented in the next subsection, which characterizes all ext-int. quas affine functions.*

5.3.2 The characterization theorem

Theorem 5.11 *Let $1 \leq k \leq n - 1$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$. The following statements are then equivalent.*

- (i) *f is ext-int. polyaffine.*

(ii) f is ext-int. quasilinear.

(iii) f is ext-int. one affine.

(iv) There exist $c_s \in \Lambda^{(k+1)s}(\mathbb{R}^n)$, $0 \leq s \leq \left\lfloor \frac{n}{k+1} \right\rfloor$ and $d_r \in \Lambda^{(n-k+1)r}(\mathbb{R}^n)$, $0 \leq r \leq \left\lfloor \frac{n}{n-k+1} \right\rfloor$ such that, for every $\xi \in \Lambda^{k+1}$, $\eta \in \Lambda^{k-1}$

$$f(\xi, \eta) = \sum_{s=0}^{\left\lfloor \frac{n}{k+1} \right\rfloor} \langle c_s; \xi^s \rangle + \sum_{r=0}^{\left\lfloor \frac{n}{n-k+1} \right\rfloor} \langle d_r; (*\eta)^r \rangle.$$

Remark 5.12 (i) The theorem is rather striking in the following respect: It says that there are no ‘new’ nonlinear ext-int. quasilinear functions (or ext-int. polyaffine or ext-int. one affine) functions, i.e knowing all ext. one affine functions and int. one affine functions are enough for knowing all the ext-int. one affine ones.

More precisely, every ext-int. polyaffine function is a sum of an ext. polyaffine function in the ‘first’ variable and an int. polyaffine function in the ‘second’ variable. In fact, even more is true. Only one of these two functions can be nonaffine. Indeed, if the ext. polyaffine function in the first variable is not affine, we must have $s(k+1) \leq n$ for some integer $s \geq 2$, since otherwise ξ^s is identically 0 for every integer $s \geq 2$. Similarly, if the int. polyaffine function in the second variable is not affine, we must have $r(n-k+1) \leq n$ for some integer $r \geq 2$, since otherwise $(*\eta)^r$ is identically 0 for every integer $r \geq 2$. But this implies,

$$\frac{1}{s} + \frac{1}{r} \geq \frac{k+1}{n} + \frac{n-k+1}{n} = \frac{n+2}{n} = 1 + \frac{2}{n} > 1,$$

but this is a contradiction since both s and r are integers and $s, r \geq 2$, we obtain,

$$\frac{1}{s} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

(ii) Note also that this is only true at the level of affine functions, but not at the level of convex ones. More precisely, every ext-int. polyconvex function need not be a sum of an ext. polyconvex function in the ‘first’ variable and an int. polyconvex function in the ‘second’ variable. The following counter example makes this clear.

Take $k = 1$ and $n \geq 4$ and consider the function $f : \Lambda^2(\mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}$, given by,

$$f(\xi, \eta) = \exp(|\xi \wedge \xi|^2 + \eta^2) \quad \text{for every } \xi \in \Lambda^2, \eta \in \mathbb{R}.$$

This function is clearly not a sum of an ext. polyconvex function in the ‘first’ variable and an int. polyconvex function in the ‘second’ variable, but is ext-int. polyconvex, though not convex. Also even if an ext-int. polyconvex function is a sum of an ext. polyconvex function in the ‘first’ variable and an int. polyconvex function in the ‘second’ variable, both can be nonlinear, as is evident in the following simple example of a function $f : \Lambda^2(\mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}$, given by,

$$f(\xi, \eta) = |\xi \wedge \xi|^2 + \eta^2 \quad \text{for every } \xi \in \Lambda^2, \eta \in \mathbb{R}.$$

Proof (i) \Rightarrow (ii) \Rightarrow (iii) follows from Theorem 5.3. (iv) \Rightarrow (i) is obvious from the definition of ext-int. polyconvexity. So we only need to prove (iii) \Rightarrow (iv). We divide the proof in four steps.

Step 1: By corollary 5.9, we obtain,

$$f(\xi, \eta) = \sum_{s=0}^{\lfloor \frac{n}{k+1} \rfloor} \langle c_s(\eta); \xi^s \rangle, \quad (5.1)$$

where $c_s(\eta) \in \Lambda^{(k+1)s}(\mathbb{R}^n)$ depends on η in such a way that the function $\eta \mapsto f(\xi, \eta)$ is int. one affine for every $\xi \in \Lambda^{k+1}(\mathbb{R}^n)$. Defining $f_s(\xi, \eta) := \langle c_s(\eta); \xi^s \rangle$, we see that due to different degrees of homogeneity in ξ , for each s , f_s must be ext-int. one affine. So it is enough to consider f_s for a fixed, but arbitrary s , $0 \leq s \leq \lfloor \frac{n}{k+1} \rfloor$.

Step 2: Now, we fix an s and write

$$f_s(\xi, \eta) = \sum_{I \in \mathcal{T}^{(k+1)s}} c_s^I(\eta) (\xi^s)_I, \quad (5.2)$$

where $c_s^I(\eta)$ and $(\xi^s)_I$ denotes the I -th component of $c_s(\eta)$ and ξ^s respectively. Now we will show that for each multiindex $I \in \mathcal{T}^{(k+1)s}$, c_s^I must be int. one affine. Clearly, there is nothing to prove if $s = 0$, so we assume $1 \leq s \leq \lfloor \frac{n}{k+1} \rfloor$. Let $I = i_1 i_2 \dots i_{(k+1)s}$. Then we define,

$$\xi_1 = e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_{k+1}} + e^{i_{k+2}} \wedge \dots \wedge e^{i_{2(k+1)}} + \dots + e^{i_{(k+1)(s-1)+1}} \wedge \dots \wedge e^{i_{(k+1)s}}.$$

Then $(\xi_1)^s = (s!)e^I$ and hence $f_s(\xi_1, \eta) = (s!)c_s^I(\eta)$. Since f_s is ext-int. one affine, c_s^I must be int. one affine by corollary 5.9.

Step 3: Since c_s^I must be int. one affine, we can write,

$$c_s^I(\eta) = \sum_{r=0}^{\lfloor \frac{n}{n-k+1} \rfloor} \langle d_{r,s}^I; (*\eta)^r \rangle. \quad (5.3)$$

and thus we can write,

$$f_s(\xi, \eta) = \sum_{I \in \mathcal{T}^{(k+1)s}} \left(\sum_{r=0}^{\lfloor \frac{n}{n-k+1} \rfloor} \langle d_{r,s}^I; (*\eta)^r \rangle \right) (\xi^s)_I. \quad (5.4)$$

$$= \sum_{r=0}^{\lfloor \frac{n}{n-k+1} \rfloor} \left(\sum_{I \in \mathcal{T}^{(k+1)s}} \langle d_{r,s}^I; (*\eta)^r \rangle (\xi^s)_I \right). \quad (5.5)$$

Once again, by different degree of homogeneity in $*\eta$, it is enough to consider fixed but arbitrary r , $0 \leq r \leq \lfloor \frac{n}{n-k+1} \rfloor$. To that effect, we define,

$$f_{r,s}(\xi, \eta) = \sum_{I \in \mathcal{T}^{(k+1)s}} \langle d_{r,s}^I; (*\eta)^r \rangle (\xi^s)_I.$$

This can be written as,

$$f_{r,s}(\xi, \eta) = \sum_{I \in \mathcal{T}^{(k+1)s}} \sum_{J \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{I,J} ((*\eta)^r)_J (\xi^s)_I \quad (5.6)$$

Step 4: To prove the claim, it is enough to prove that $d_{r,s}^{I,J} = 0$ for all $I \in \mathcal{T}^{(k+1)s}$, $J \in \mathcal{T}^{(n-k+1)r}$ unless $rs = 0$. We now proceed to show that.

Let $1 \leq s \leq \left\lfloor \frac{n}{k+1} \right\rfloor$, $1 \leq r \leq \left\lfloor \frac{n}{n-k+1} \right\rfloor$. First note that this implies, for any $I \in \mathcal{T}^{(k+1)s}$, $J \in \mathcal{T}^{(n-k+1)r}$, there is at least one common index between I and J (In fact, there must be at least two). Let $I = \{i_1 i_2 \dots i_{(k+1)s}\}$ and $J = \{j_1 j_2 \dots j_{(n-k+1)r}\}$ and $i_p = j_q$ for some p, q . To keep the presentation cleaner, we need to adopt a few shorthands here.

Notation 5.13 We divide the multiindex I into s blocks of multiindices, each containing $k+1$ indices as follows: I^α , $\alpha = 1, 2, \dots, s$, will denote the α -th block of $k+1$ indices, starting from the first, i.e starting from i_1 . For example, for $\alpha = 1$, $I^1 = \{i_1 i_2 \dots i_{k+1}\}$ and for $\alpha = 2$, $I^2 = \{i_{k+2} i_{k+3} \dots i_{2(k+1)}\}$ and so on. More precisely, $I^\alpha = \{i_{(\alpha-1)(k+1)+1} i_{(\alpha-1)(k+1)+2} \dots i_{\alpha(k+1)}\}$ for all $1 \leq \alpha \leq s$ integer. Similarly, we divide the multiindex J into r blocks of multiindices, each containing $n-k+1$ indices as follows: J^β , $\beta = 1, 2, \dots, r$, will denote the β -th block of $n-k+1$ indices, starting from the first, i.e starting from j_1 . For example, for $\beta = 1$, $J^1 = \{j_1 j_2 \dots j_{n-k+1}\}$ and for $\beta = 2$, $J^2 = \{j_{n-k+2} j_{n-k+3} \dots j_{2(n-k+1)}\}$ and so on. More precisely, $J^\beta = \{j_{(\beta-1)(n-k+1)+1} j_{(\beta-1)(n-k+1)+2} \dots j_{\beta(n-k+1)}\}$ for all $1 \leq \beta \leq r$ integer.

Also, for the sake of clarity, let $I_p \in \mathcal{T}^{k+1}$ denote the block of $(k+1)$ indices of I which contains i_p and $J_q \in \mathcal{T}^{n-k+1}$ denote the block of $(n-k+1)$ indices of J which contains j_q . Note that in our notation, this implies, $I_p = I^{\left\lfloor \frac{p-1}{k+1} \right\rfloor + 1}$ and $J_q = J^{\left\lfloor \frac{q-1}{n-k+1} \right\rfloor + 1}$. Also, let $I'_p = I_p \setminus \{i_p\}$ and $J'_q = J_q \setminus \{j_q\}$.

Now we choose,

$$\left\{ \begin{array}{l} a = e^{i_p} = e^{j_q}, \\ b = e^{I'_p} + (*e^{J'_q}), \\ \xi = \frac{1}{(s-1)!} \sum_{\substack{1 \leq \alpha \leq s \\ \alpha \neq \left\lfloor \frac{p-1}{k+1} \right\rfloor + 1}} e^{I^\alpha}, \\ * \eta = \frac{1}{(r-1)!} \sum_{\substack{1 \leq \beta \leq r \\ \beta \neq \left\lfloor \frac{q-1}{n-k+1} \right\rfloor + 1}} e^{J^\beta}. \end{array} \right.$$

Of course, if $s = 1$, we choose $\xi = 0$ and if $r = 1$, we choose $*\eta = 0$.

Here we will disregard questions of signs, as it is unimportant for the argument and use \pm to denote that either sign is possible. Clearly,

$$\begin{aligned} a \wedge b &= e^{i_p} \wedge e^{I'_p} + e^{j_q} \wedge (*e^{J'_q}) \\ &= \pm e^{I_p}, \end{aligned} \quad (5.7)$$

and

$$\begin{aligned}
a \wedge (*b) &= a \wedge \left(*e^{I'_p} + * \left(*e^{J'_q} \right) \right) \\
&= a \wedge \left(*e^{I'_p} \right) \pm a \wedge e^{J'_q} \\
&= e^{i_p} \wedge \left(*e^{I'_p} \right) \pm e^{j_q} \wedge e^{J'_q} \\
&= \pm e^{J_q}.
\end{aligned} \tag{5.8}$$

We also have,

$$(\xi)^{s-1} = e^{I \setminus I_p}, \tag{5.9}$$

$$(*\eta)^{r-1} = e^{J \setminus J_q}. \tag{5.10}$$

Note that here we implicitly used the following facts: if $s = 1$ or 2 , the formula for $(\xi)^{s-1}$ is trivially true and if $s \geq 2$, then $k+1$ must be even, since otherwise terms containing ξ^s are absent from the expression for f . If $k+1$ is even, the formula for $(\xi)^{s-1}$ holds for any $2 \leq s \leq \left\lfloor \frac{n}{k+1} \right\rfloor$. Similarly, if $r = 1$ or 2 , the formula for $(*\eta)^{r-1}$ is trivially true and if $r \geq 2$, then $n-k+1$ must be even, since otherwise terms containing $(*\eta)^r$ are absent from the expression for f . If $n-k+1$ is even, the formula for $(*\eta)^{r-1}$ holds for any $2 \leq r \leq \left\lfloor \frac{n}{n-k+1} \right\rfloor$.

From 5.6, we have, for any $t \in [0, 1]$,

$$\begin{aligned}
f_{r,s}(\xi + ta \wedge b, \eta + ta \lrcorner b) &= \sum_{K \in \mathcal{T}^{(k+1)s}} \sum_{L \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{K,L} ((*\eta + ta \lrcorner b))^r_L ((\xi + ta \wedge b))^s_K \\
&= \sum_{K \in \mathcal{T}^{(k+1)s}} \sum_{L \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{K,L} ((*\eta \pm ta \wedge (*b))^r_L ((\xi + ta \wedge b))^s_K.
\end{aligned} \tag{5.11}$$

The term which is quadratic in t in the above expression on the right hand side is,

$$\pm t^2 (r!)(s!) \sum_{K \in \mathcal{T}^{(k+1)s}} \sum_{L \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{K,L} \left((*\eta)^{r-1} \wedge a \wedge (*b) \right)_L \left(\xi^{s-1} \wedge a \wedge b \right)_K.$$

Now with our choice of a, b, ξ, η , this becomes,

$$\begin{aligned}
&\pm t^2 (r!)(s!) \sum_{K \in \mathcal{T}^{(k+1)s}} \sum_{L \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{K,L} \left(e^{J \setminus J_q} \wedge (\pm e^{J_q}) \right)_L \left(e^{I \setminus I_p} \wedge (\pm e^{I_p}) \right)_K, \\
&= \pm t^2 (r!)(s!) \sum_{K \in \mathcal{T}^{(k+1)s}} \sum_{L \in \mathcal{T}^{(n-k+1)r}} d_{r,s}^{K,L} (\pm e^J)_L (\pm e^I)_K \\
&= \pm t^2 (r!)(s!) d_{r,s}^{I,J}.
\end{aligned}$$

Now, since $f_{r,s}$ must be ext-int. one affine, $f_{r,s}(\xi + ta \wedge b, \eta + ta \lrcorner b)$ must be affine in t , which forces $d_{r,s}^{I,J} = 0$ and completes the proof. ■

5.4 The ext-int convexity properties and the classical notions of convexity.

5.4.1 The projection maps

As in the third chapter, it is also possible here to point out the relationship between the notions introduced in this chapter and the classical notions of the calculus of variations namely rank one convexity, quasiconvexity and polyconvexity (see [25]). We first introduce some notations. As usual, by abuse of notations, we identify $\Lambda^k(\mathbb{R}^n)$ with $\mathbb{R}^{\binom{n}{k}}$.

Notation 5.14 Let $1 \leq k \leq n-1$. To a matrix $\Xi \in \mathbb{R}^{\binom{n}{k} \times n}$, the upper indices being ordered alphabetically, written, depending on the context, as

$$\begin{aligned} \Xi &= \begin{pmatrix} \Xi_1^{1 \dots (k)} & \dots & \Xi_n^{1 \dots (k)} \\ \vdots & \ddots & \vdots \\ \Xi_1^{(n-k+1) \dots n} & \dots & \Xi_n^{(n-k+1) \dots n} \end{pmatrix} \\ &= (\Xi_i^I)_{i \in \{1, \dots, n\}}^{I \in \mathcal{T}_k^n} = \begin{pmatrix} \Xi_1^{1 \dots (k)} \\ \vdots \\ \Xi_n^{(n-k+1) \dots n} \end{pmatrix} = (\Xi_1, \dots, \Xi_n) \end{aligned}$$

we associate a map $\pi^{\text{ext-int}, k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ in the following way

$$\pi^{\text{ext-int}, k}(\Xi) = \left(\pi^{\text{ext}, k+1}(\Xi), \pi^{\text{int}, k-1}(\Xi) \right)$$

where $\pi^{\text{ext}, k+1}$ and $\pi^{\text{int}, k-1}$ are as defined in chapter 3.

We now list some useful properties of this map.

Theorem 5.15 For any $1 \leq k \leq n-1$, The map $\pi^{\text{ext-int}, k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ as defined above is surjective.

Proof We need to show, given $\alpha \in \Lambda^{k+1}(\mathbb{R}^n)$ and $\beta \in \Lambda^{k-1}(\mathbb{R}^n)$, there exist a matrix $\Xi \in \mathbb{R}^{\binom{n}{k} \times n}$ such that $\pi^{\text{ext-int}, k}(\Xi) = (\alpha, \beta)$.

Observe first that by linearity of the maps $\pi^{\text{ext}, k+1}$ and $\pi^{\text{int}, k-1}$, we can assume, without loss of generality that α is ext. one-decomposable and β is int. one-decomposable. Hence by lemma 5.5, there exist $a \in \Lambda^1, b \in \Lambda^k$ such that $\alpha = a \wedge b$ and $a \lrcorner b = 0$. Considering a as a vector in \mathbb{R}^n and b as a vector in $\mathbb{R}^{\binom{n}{k}}$, we define $\Xi_1 = a \otimes b$. Again similarly, by lemma 5.5, there exist $c \in \Lambda^1, d \in \Lambda^k$ such that $\beta = c \lrcorner d$ and $c \wedge d = 0$. Considering c as a vector in \mathbb{R}^n and d as a vector in $\mathbb{R}^{\binom{n}{k}}$, we define $\Xi_2 = c \otimes d$. Finally, we set $\Xi = \Xi_1 + \Xi_2$.

Now since $\pi^{\text{ext}, k+1}(a \otimes b) = a \wedge b$ and $\pi^{\text{int}, k-1}(a \otimes b) = a \lrcorner b$, we have,

$$\begin{aligned} \pi^{\text{ext}, k+1}(\Xi_1) &= \alpha & ; & & \pi^{\text{int}, k-1}(\Xi_1) &= 0, \\ \pi^{\text{ext}, k+1}(\Xi_2) &= 0 & ; & & \pi^{\text{int}, k-1}(\Xi_2) &= \beta. \end{aligned}$$

Hence, we have,

$$\pi^{\text{ext-int}, k}(\Xi) = \pi^{\text{ext}, k+1}(\Xi) + \pi^{\text{int}, k-1}(\Xi) = \alpha,$$

and

$$\pi^{\text{int},k-1}(\Xi) = \pi^{\text{int},k-1}(\Xi_1) + \pi^{\text{int},k-1}(\Xi_2) = \beta.$$

This completes the proof. ■

The following properties are immediate from the properties of $\pi^{\text{ext},k+1}$ and $\pi^{\text{int},k-1}$.

Proposition 5.16 *Let $1 \leq k \leq n-1$ and $\pi^{\text{ext-int},k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ be as above.*

(i) *If $\alpha \in \Lambda^1(\mathbb{R}^n) \sim \mathbb{R}^n$ and $\beta \in \Lambda^k(\mathbb{R}^n) \sim \mathbb{R}^{\binom{n}{k}}$ then,*

$$\pi^{\text{ext-int},k}(\alpha \otimes \beta) = (\alpha \wedge \beta, \alpha \lrcorner \beta).$$

(ii) *Let $\omega \in C^1(\Omega; \Lambda^k)$, then, by abuse of notations,*

$$\pi^{\text{ext-int},k}(\nabla \omega) = (d\omega, \delta \omega).$$

Note that Proposition 5.16 immediately implies the following.

Theorem 5.17 *Let $1 \leq k \leq n-1$,*

$$f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \text{and} \quad \pi^{\text{ext-int},k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$$

be the projection map. Then the following equivalences hold

$$f \text{ ext-int. one convex} \Leftrightarrow f \circ \pi^{\text{ext-int},k} \text{ rank one convex}$$

$$f \text{ ext-int. quasiconvex} \Leftrightarrow f \circ \pi^{\text{ext-int},k} \text{ quasiconvex}.$$

Proof With proposition 5.16 at our disposal, the proof is exactly like the proof of conclusion (i) and (ii) of Theorem 3.54. ■

We are however, at present, unable to prove the analogue of the third conclusion of Theorem 3.54. It appears that it would be possible to prove this by adapting the same strategy we employed to prove statement (iii) of Theorem 3.54. We can also anticipate that the analogue of Lemma 3.47 would be true in this setting too. However, the proof of the lemma was already complicated in the ext. polyconvexity case, but in this case it is going to be even more, quite possibly considerably more tedious to prove such a lemma. We leave this result as a conjecture.

Conjecture 5.18 *Let $1 \leq k \leq n-1$,*

$$f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R} \quad \text{and} \quad \pi^{\text{ext-int},k} : \mathbb{R}^{\binom{n}{k} \times n} \rightarrow \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$$

be the projection map. Then

$$f \text{ ext-int. polyconvex} \Leftrightarrow f \circ \pi^{\text{ext-int},k} \text{ polyconvex}$$

5.5 Existence theorems for minimization problems

We now conclude this chapter with a few existence theorems for minimization problems involving such functionals. The main point, as we already remarked in the introduction to this chapter, is that this type of functionals are actually coercive, due to Gaffney inequality.

5.5.1 Existence theorems without lower order terms

We start with two existence theorems for minimization problem for ext-int. quasiconvex functions. The proof of both of them being very similar, we shall only prove the first one.

Theorem 5.19 *Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^k)$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext-int. quasiconvex verifying, for every $\xi \in \Lambda^{k+1}, \eta \in \Lambda^{k-1}$,*

$$c_1 (|\xi|^p + |\eta|^p - 1) \leq f(\xi, \eta) \leq c_2 (|\xi|^p + |\eta|^p + 1) \quad (5.12)$$

for some $c_1, c_2 > 0$. Let

$$(\mathcal{P}_T) \quad \inf \left\{ \int_{\Omega} f(d\omega, \delta\omega) : \omega \in \omega_0 + W_T^{1,p}(\Omega; \Lambda^k) \right\} = m.$$

Then the problem (\mathcal{P}_T) has a minimizer.

Proof Let $\{\omega_s\}$ be a minimizing sequence. Then by the growth condition 5.12, we find that there exists a constant $c > 0$ such that,

$$\|d\omega_s\|_{L^p} + \|\delta\omega_s\|_{L^p} \leq c. \quad (5.13)$$

By corollary 2.41, we see that there exists a constant $c_1 > 0$ such that,

$$\|\omega_s\|_{W^{1,p}} \leq c_1. \quad (5.14)$$

Thus $\{\omega_s\}$ is uniformly bounded in $W^{1,p}$ and hence there exists $\omega \in W^{1,p}$ such that $\omega_s \rightharpoonup \omega$ in $W^{1,p}$.

Since for every $(\xi, \eta) \in \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ the function $(\xi, \eta) \mapsto f(\xi, \eta)$ is ext-int. quasiconvex implies that the function $\Xi \mapsto f(\pi^{\text{ext-int}, k}(\Xi))$ is quasiconvex for every $\Xi \in \mathbb{R}^{\binom{n}{k} \times n}$, we have by classical semicontinuity result (see Theorem 8.11 in [25]),

$$\begin{aligned} m &= \liminf_{s \rightarrow \infty} \int_{\Omega} f(d\omega_s, \delta\omega_s) = \liminf_{s \rightarrow \infty} \int_{\Omega} f\left(\pi^{\text{ext-int}, k}(\nabla\omega_s)\right) \geq \int_{\Omega} f\left(\pi^{\text{ext-int}, k}(\nabla\omega)\right) \\ &= \int_{\Omega} f(d\omega, \delta\omega) \geq m. \end{aligned}$$

Note that $\omega_s \rightharpoonup \omega$ in $W^{1,p}$ implies $\nu \wedge \omega_s \rightarrow \nu \wedge \omega$ in $W^{1-\frac{1}{p}, p}(\partial\Omega)$. This completes the proof. ■

Similarly we can prove the following.

Theorem 5.20 Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^k)$ and $f : \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be ext-int. quasiconvex verifying, for every $\xi \in \Lambda^{k+1}, \eta \in \Lambda^{k-1}$,

$$c_1 (|\xi|^p + |\eta|^p - 1) \leq f(\xi, \eta) \leq c_2 (|\xi|^p + |\eta|^p + 1)$$

for some $c_1, c_2 > 0$. Let

$$(\mathcal{P}_N) \quad \inf \left\{ \int_{\Omega} f(d\omega, \delta\omega) : \omega \in \omega_0 + W_N^{1,p}(\Omega; \Lambda^k) \right\} = m.$$

Then the problem (\mathcal{P}_N) has a minimizer.

5.5.2 Existence theorems with lower order terms

The case with lower order terms is essentially the same.

Theorem 5.21 Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^k)$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \times \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function satisfying for almost every $x \in \Omega$, for every $(\omega, \xi, \eta) \in \Lambda^k(\mathbb{R}^n) \times \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$,

$(\xi, \eta) \mapsto f(x, \omega, \xi, \eta)$ is ext-int. quasiconvex,

$$\alpha_1 (|\xi|^p + |\eta|^p) + \beta_1 |\omega|^q + \gamma_1(x) \leq f(x, \omega, \xi, \eta) \leq \alpha_2 (|\xi|^p + |\eta|^p) + \beta_2 |\omega|^r + \gamma_2(x) \quad (5.15)$$

where $\alpha_2 \geq \alpha_1 > 0, \beta_1 \in \mathbb{R}, \beta_2 \geq 0, \gamma_1, \gamma_2 \in L^1(\Omega), p > q \geq 1$ and $1 \leq r \leq np/(n-p)$ if $p < n$ and $1 \leq r < \infty$ if $p \geq n$.

Let

$$(\mathcal{P}_T) \quad \inf \left\{ \int_{\Omega} f(x, \omega, d\omega, \delta\omega) : \omega \in \omega_0 + W_T^{1,p}(\Omega; \Lambda^k) \right\} = m.$$

Then the problem (\mathcal{P}_T) has a minimizer.

Proof Let $\{\omega_s\}$ be a minimizing sequence. Then by the growth condition 5.15, we have for s sufficiently large,

$$m+1 \geq \alpha_1 (\|d\omega_s\|_{L^p}^p + \|\delta\omega_s\|_{L^p}^p) - |\beta_1| \|\omega_s\|_{L^q}^q - \|\gamma_1(x)\|_{L^1}.$$

Since by Hölder inequality, we have $\|\omega_s\|_{L^q}^q \leq |\Omega|^{\frac{p-q}{p}} \|\omega_s\|_{L^p}^q$, we deduce that we can find constants $c_1, c_2 > 0$ such that,

$$\begin{aligned} m+1 &\geq \alpha_1 (\|d\omega_s\|_{L^p}^p + \|\delta\omega_s\|_{L^p}^p) - c_1 \|\omega_s\|_{L^p}^q - c_2 \\ &\geq \alpha_1 (\|d\omega_s\|_{L^p}^p + \|\delta\omega_s\|_{L^p}^p) - c_1 \|\omega_s\|_{W^{1,p}}^q - c_2 \end{aligned}$$

By corollary 2.41, we see that there exists constants $c_3, c_4, c_5 > 0$ such that,

$$m+1 \geq c_3 \|\omega_s\|_{W^{1,p}}^p - c_4 \|\omega_0\|_{W^{1,p}}^p - c_1 \|\omega_s\|_{W^{1,p}}^q - c_5$$

and hence for some $c_6 > 0$,

$$m + 1 \geq c_3 \|\omega_s\|_{W^{1,p}}^p - c_1 \|\omega_s\|_{W^{1,p}}^q - c_6. \quad (5.16)$$

This implies that $\{\omega_s\}$ is uniformly bounded in $W^{1,p}$, i.e there exists a constant $c > 0$ such that,

$$\|\omega_s\|_{W^{1,p}} \leq c. \quad (5.17)$$

To see that this is indeed the case, suppose $\{\omega_s\}$ is not uniformly bounded in $W^{1,p}$, then this implies there exists a subsequence $\{\omega_{s_i}\}$ such that $\|\omega_{s_i}\|_{W^{1,p}} \geq i$ for every $i \in \mathbb{N}$. But since $p > q$, there exists an integer $i_0 \in \mathbb{N}$ such that

$$m + 1 < c_3 x^p - c_1 x^q - c_6$$

for every real number $x \geq i_0$. But this implies

$$m + 1 < c_3 \|\omega_{s_{i_0}}\|_{W^{1,p}}^p - c_1 \|\omega_{s_{i_0}}\|_{W^{1,p}}^q - c_6,$$

which contradicts (5.16).

Hence $\{\omega_s\}$ is uniformly bounded in $W^{1,p}$ and thus there exists $\omega \in W^{1,p}$ such that $\omega_s \rightharpoonup \omega$ in $W^{1,p}$.

Since for almost every $x \in \Omega$, for every $(\omega, \xi, \eta) \in \Lambda^k(\mathbb{R}^n) \times \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$ the function $(\xi, \eta) \mapsto f(x, \omega, \xi, \eta)$ is ext-int. quasiconvex implies that the function $\Xi \mapsto f(x, \omega, \pi^{\text{ext-int}, k}(\Xi))$ is quasiconvex for almost every $x \in \Omega$, for every $(\omega, \Xi) \in \mathbb{R}^{\binom{n}{k}} \times \mathbb{R}^{\binom{n}{k} \times n}$, we have by classical semicontinuity result (see Theorem 8.11 in [25]),

$$\begin{aligned} m &= \liminf_{s \rightarrow \infty} \int_{\Omega} f(x, \omega_s, d\omega_s, \delta\omega_s) \\ &= \liminf_{s \rightarrow \infty} \int_{\Omega} f\left(x, \omega_s, \pi^{\text{ext-int}, k}(\nabla\omega_s)\right) \\ &\geq \int_{\Omega} f\left(x, \omega, \pi^{\text{ext-int}, k}(\nabla\omega)\right) \\ &= \int_{\Omega} f(x, \omega, d\omega, \delta\omega) \\ &\geq m. \end{aligned}$$

This completes the proof since $\omega_s \rightharpoonup \omega$ in $W^{1,p}$ implies $\nu \wedge \omega_s \rightarrow \nu \wedge \omega$ in $W^{1-\frac{1}{p}, p}(\partial\Omega)$, ensuring $\nu \wedge \omega = \nu \wedge \omega_0$. ■

Theorem 5.22 *Let $1 \leq k \leq n-1$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set, $\omega_0 \in W^{1,p}(\Omega; \Lambda^k)$ and $f : \Omega \times \Lambda^k(\mathbb{R}^n) \times \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a Carathéodory function satisfying for almost every $x \in \Omega$, for every $(\omega, \xi, \eta) \in \Lambda^k(\mathbb{R}^n) \times \Lambda^{k+1}(\mathbb{R}^n) \times \Lambda^{k-1}(\mathbb{R}^n)$,*

$$\begin{aligned} &(\xi, \eta) \mapsto f(x, \omega, \xi, \eta) \text{ is ext-int. quasiconvex,} \\ &\alpha_1 (|\xi|^p + |\eta|^p) + \beta_1 |\omega|^q + \gamma_1(x) \leq f(x, \omega, \xi, \eta) \leq \alpha_2 (|\xi|^p + |\eta|^p) + \beta_2 |\omega|^r + \gamma_2(x) \end{aligned} \quad (5.18)$$

where $\alpha_2 \geq \alpha_1 > 0, \beta_1 \in \mathbb{R}, \beta_2 \geq 0, \gamma_1, \gamma_2 \in L^1(\Omega), p > q \geq 1$ and $1 \leq r \leq np/(n-p)$ if $p < n$ and $1 \leq r < \infty$ if $p \geq n$.

Let

$$(\mathcal{P}_N) \quad \inf \left\{ \int_{\Omega} f(x, \omega, d\omega, \delta\omega) : \omega \in \omega_0 + W_N^{1,p}(\Omega; \Lambda^k) \right\} = m.$$

Then the problem (\mathcal{P}_N) has a minimizer.

Remark 5.23 *It is clear that these theorems will continue to hold for non-contractible domains if the spaces are replaced by $\omega_0 + W_T^{1,p}(\Omega; \Lambda^k) \cap \mathcal{H}_T^{\perp}(\Omega; \Lambda^k)$, instead of $\omega_0 + W_T^{1,p}(\Omega; \Lambda^k)$ in theorem 5.21 and by $\omega_0 + W_N^{1,p}(\Omega; \Lambda^k) \cap \mathcal{H}_N^{\perp}(\Omega; \Lambda^k)$, instead of $\omega_0 + W_N^{1,p}(\Omega; \Lambda^k)$ in theorem 5.22.*

Part II

Some Boundary value problems for Differential Forms

Foreword to part II

The equation

$$\operatorname{div}(A(x)(\nabla u)) = f,$$

for an unknown function u has played a central role in the theory of elliptic partial differential equations. In dimension 3, the equation

$$\operatorname{curl}(A(x)(\operatorname{curl} E)) = f,$$

for an unknown vector field E is called the time-harmonic Maxwell's equation. Both these equations can be seen as special cases of the following general equation

$$\delta(A(x)(d\omega)) = f,$$

for a differential k -form ω , where $0 \leq k \leq n - 1$. We call this operator the *linear Maxwell equation for k -forms*. In the same spirit, we call the equations

$$\delta(A(x, d\omega)) = f$$

and

$$\delta(A(x)(d\omega)) = f(\omega),$$

the *quasilinear Maxwell equation for k -forms* and *semilinear Maxwell equation for k -forms* respectively.

We are going to study some boundary value problems for the linear, semilinear and quasilinear Maxwell equations for k -forms in an open, smooth, bounded and contractible domain $\Omega \subset \mathbb{R}^n$. Existence results, interior regularity results in $W^{r,p}$ and $C^{r,\alpha}$ spaces and up to the boundary regularity results in $W^{r,2}$ spaces are obtained for full Dirichlet boundary data problem

$$\begin{cases} \delta(A(x)(d\omega)) = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega, \end{cases}$$

and the related second order elliptic system, when $\lambda \in \mathbb{R}$,

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

These results yield the corresponding results for the dual problems

$$\begin{cases} d(A(x)(\delta\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d(A(x)(\delta\omega)) = \lambda\omega + f \text{ in } \Omega, \\ d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

The up to the boundary regularity results in $W^{r,2}$ spaces also enables us to solve the following two first order systems with optimal regularity in $W^{r,2}$.

$$\begin{cases} d(A(x)\omega) = f \quad \text{and} \quad \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \wedge A(x)\omega = \nu \wedge \omega_0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d(A(x)\omega) = f \quad \text{and} \quad \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \lrcorner B(x)\omega = \nu \lrcorner \omega_0 & \text{on } \partial\Omega. \end{cases}$$

For both these systems, under reasonable assumption on the coefficient $A(x)$ and $B(x)$, we can show the existence of a solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k)$, assuming ω_0 to be $W^{r+2,2}$ and f, g to be $W^{r,2}$. This also yields the optimal $W^{r,2}$ regularity result for the Hodge-type system

$$\begin{cases} \delta(A(x)(d\omega)) + \delta d\omega = \lambda\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega, \\ \nu \wedge \delta\omega = 0 \text{ on } \partial\Omega. \end{cases}$$

We show existence results for two types of semilinear problems. The sign of the semilinearity is crucial for these problems. When the energy functional is coercive, we can solve the following prototype problem

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + |\omega|^{p-2}\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega, \end{cases}$$

for any $f \in L^{p'}$, where p' is the Hölder conjugate exponent of p , and for any nonnegative λ to the right of the spectrum of the linear principal part.

When the sign of the semilinearity makes the energy functional indefinite, we show the existence for the eigenvalue problem

$$\begin{cases} \delta(A(x)(d\omega)) + |\omega|^{p-2}\omega = \lambda\omega & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

by using a Nehari-Pankov manifold technique.

The quasilinear case, in a sense, is very similar to the linear theory. We use monotone operator theory to show the existence results for the full Dirichlet boundary data problem

$$\begin{cases} \delta(A(x, d\omega)) = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega, \end{cases}$$

and the related quasilinear elliptic system

$$\begin{cases} \delta(A(x, d\omega)) = f & \text{in } \Omega, \\ \delta\omega = 0 & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The material in this part is divided into two chapters. In chapter 6, we are going to treat the linear case. Existence results, regularity results and its consequences. Chapter 7 deals with the existence theory for the semilinear and quasilinear cases.

Chapter 6

Maxwell operator for k -forms: Linear theory

6.1 Introduction

We are interested in the boundary value problems involving the following Maxwell type operator on k -forms:

$$\delta(A(x)(d\omega)) = f \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is an open, smooth, bounded set. When $k = 0$, i.e ω is a real valued function, this equation is the familiar,

$$\operatorname{div}(A(x)(\nabla\omega)) = f.$$

When $A(x) \equiv \mathbf{I}$, the $n \times n$ identity matrix, this is just the scalar Poisson equation,

$$\Delta\omega = f.$$

Also, when $k = 1$, i.e ω is an 1-form and hence can be identified with a vector field, in three dimensions ($n = 3$) this reduces to, up to a sign,

$$\operatorname{curl}(A(x)(\operatorname{curl}\omega)) = f,$$

which is the principal part of the time harmonic Maxwell equation.

In the forthcoming analysis $\Omega \subset \mathbb{R}^n$ will always be assumed to be open, bounded, smooth and contractible. We shall not concern ourselves with the question of optimal smoothness requirements on the boundary. Also, the contractibility hypothesis can be dropped with the obvious modifications to the results presented, but we refrain from doing so to keep the presentation simpler. Our primary concern is to deduce an existence theorem (cf. theorem 6.32) regarding the solvability of the following boundary value problem:

$$\begin{cases} \delta(A(x)(d\omega)) = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega, \end{cases} \quad (6.1)$$

when $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ is sufficiently smooth. This result is new and as far as we are aware, the question of solvability of the boundary value problem with prescribed full Dirichlet data has not been investigated so far, even for 1-forms. The remarkable feature of this problem

is that this boundary value problem is very far from being elliptic. In fact, it is quite clear a priori, that the solution space, must be infinite dimensional if it is non-empty. Throughout the first few sections of this chapter this lack of ellipticity and as a consequence, the lack of Fredholm property will be of crucial importance.

The proof of this result can be approached in two essentially equivalent ways, up to a slight sharpening or weakening of the hypotheses. When A is symmetric this problem has a variational structure and under slightly stronger ellipticity hypothesis on A (the *Legendre* condition, cf. Definition 6.2), existence can be deduced by using direct methods as developed in part 1 (cf. theorem 3.64). However, we take the more direct route here which enables us to drop the symmetry assumption and also permits us to use weaker ellipticity condition (the *Legendre-Hadamard* condition, cf. Definition 6.1) on A .

To prove such a result, we need to investigate the following related boundary value problem:

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (6.2)$$

We show that this problem is well-posed, elliptic and has the Fredholm property in the scale of $W^{r,2}$ spaces for any k and any n .

In $k = 1$ and $n = 3$, this problem is the prototype for the well-studied problem,

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl} \vec{\omega}) = k_0^2 \varepsilon \vec{\omega} + \vec{j} \text{ in } \Omega, \\ \operatorname{div}(\varepsilon \vec{\omega}) = 0 \text{ in } \Omega, \\ \vec{n} \times \vec{\omega} = 0 \text{ on } \partial\Omega. \end{cases}$$

This is the time-harmonic Maxwell's equation for an electric field inside a cavity of a perfectly conducting material. Of course, interchanging the constants μ and ε , i.e the permeability and permittivity of the medium inside the cavity leads to the time-harmonic Maxwell's equation for the magnetic field with impedance boundary conditions. There is a large amount of literature in physics, engineering and mathematics regarding this problem. There are a number of articles and even books where results concerning existence and regularity of solutions to the time-harmonic Maxwell's equation or some of its variants have been shown (cf. [42], [45], [51] and references therein). For the particular case of 1-forms in 3 dimensions, the most general results available in the literature seems to be concerning the corresponding equations for an anisotropic inhomogeneous medium inside the cavity, for which the equation is of the form,

$$\begin{cases} \operatorname{curl}(\mu(x)^{-1}(\operatorname{curl} \vec{\omega})) = k_0^2 \vec{\omega} + \vec{j} \text{ in } \Omega, \\ \operatorname{div}(\vec{\omega}) = 0 \text{ in } \Omega, \\ \vec{n} \times \vec{\omega} = 0 \text{ on } \partial\Omega. \end{cases}$$

The usual assumption in the literature is $\mu(x)$ is symmetric and uniformly positive definite (see e.g [44] and references therein).

The result presented in this chapter for the equation (6.2) is a generalization to the case of k -form in arbitrary space dimensions n , where we assumed neither the symmetry assumption nor the uniform positive definiteness, which is replaced by Legendre-Hadamard type of ellipticity assumption. The result, in this generality, as far as we are aware, has not been treated elsewhere. Indeed, in theorem 6.11, we show that existence and spectral theory for (6.2) is possible under reasonably minimal hypotheses. The existence result for (6.1) is derived from Theorem 6.11. We also show in theorem 6.30 that a full elliptic regularity theory in the scale of $W^{r,2}$ spaces is true for this system. This up to the boundary regularity estimates in $W^{r,2}$ spaces is also new in this generality. The only cases where up to the boundary regularity estimates exist in the literature are the case of the Hodge Laplacian, i.e when $A(x) \equiv \mathbf{I}$, and the case of the time-harmonic Maxwell's equation, i.e when $k = 1$ and $n = 3$. The usual methods for regularity estimates for the time-harmonic Maxwell's equation can treat fairly general matrix $A(x)$, but is restricted to 1-forms in dimension 3 alone and can not be generalized neither to any dimension nor to any k -form, although a recent argument by Dacorogna-Gangbo-Kneuss [27] seems to work in any dimension as long as $k = 1$. On the other hand, the regularity results for the Hodge Laplacian holds for any k -forms in any dimension n , but these results crucially use the fact that $A(x) \equiv \mathbf{I}$. Note also that $C^{1,\alpha}$ regularity estimates of Hamburger [35] for the quasilinear case can imply $C^{1,\alpha}$ boundary estimates for the case of Hodge Laplacian only, but not about the more general linear system (6.2) if A is not a constant multiple of identity matrix, even when $f = 0$ and $\lambda = 0$.

However, in this thesis we obtain only $W^{r,2}$ estimates up to the boundary, leaving regularity estimates in the scale of $W^{r,p}(p \neq 2)$ and $C^{r,\alpha}$ spaces to the future (see [60]).

6.2 Existence of weak solutions

We shall start by collecting the ellipticity conditions that we shall use throughout the chapter below.

Definition 6.1 A map $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ is said to satisfy the **Legendre-Hadamard condition** if A satisfies, for all $x \in \Omega$,

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma > 0$.

Definition 6.2 A map $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ is said to satisfy the **Legendre condition** if A satisfies, for all $x \in \Omega$,

$$\langle A(x)\xi ; \xi \rangle \geq \gamma |\xi|^2, \quad \text{for every } \xi \in \Lambda^{k+1}$$

for some constant $\gamma > 0$.

Along with the usual Sobolev spaces $W^{1,p}(\Omega; \Lambda^k)$ and $W_0^{1,p}(\Omega; \Lambda^k)$, we shall be using the partial Sobolev spaces $W^{d,2}(\Omega; \Lambda^k)$ and the space $W_0^{d,2}(\Omega; \Lambda^k) = W_T^{d,2}(\Omega; \Lambda^k)$, defined earlier,

quite a lot. Also consider the following subspace $W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \subset W_T^{d,2}(\Omega; \Lambda^k)$ defined by,

$$W_{\delta,T}^{d,2}(\Omega; \Lambda^k) := \{\omega \in W_T^{d,2}(\Omega; \Lambda^k); \delta\omega = 0\},$$

where the condition $\delta\omega = 0$ is understood in the sense of distributions. Clearly $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is a closed subspace of $W_T^{d,2}(\Omega; \Lambda^k)$. Also, $dW_0^{1,2}(\Omega; \Lambda^k)$ is a closed subspace of $W_T^{d,2}(\Omega; \Lambda^k)$ and

$$W_T^{d,2}(\Omega; \Lambda^k) = W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \oplus dW_0^{1,2}(\Omega; \Lambda^k).$$

(cf. theorem 2.52 for the proof of the above decomposition and section 2.5 for related results). The direct sum decomposition is clearly also orthogonal with respect to the inner product. Also note that $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ embeds continuously in $W^{1,2}$ and hence by Rellich's theorem, $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ embeds compactly in L^2 . Hence the norm $\|v\|_{W_{\delta,T}^{d,2}(\Omega; \Lambda^k)} = \|dv\|_{L^2}$ is an equivalent norm on $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$.

6.2.1 Existence in $W_{\delta,T}^{d,2}$

We start by proving a Gårding type inequality,

Theorem 6.3 *Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfy either the Legendre-Hadamard condition and is uniformly continuous or the Legendre condition and is bounded and measurable. Also let $B \in L^\infty(\Omega; L(\Lambda^k, \Lambda^{k+1}))$ and $C \in L^\infty(\Omega; L(\Lambda^k, \Lambda^k))$. Then there exist constants $\lambda_0 > 0$ and λ_1 such that,*

$$a(u, u) = \int_{\Omega} [\langle A(x)du, du \rangle + \langle B(x)u, du \rangle + \langle C(x)u, u \rangle] \geq \lambda_0 \|du\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2, \quad (6.3)$$

for all $u \in W_T^{d,2}(\Omega, \Lambda^k)$.

Remark 6.4 *Note that we need the hypotheses of uniform continuity in the case of Legendre-Hadamard condition. As is well known, even for the classical elliptic systems, the hypothesis of uniform continuity is crucial to obtain Gårding type inequality. Such an inequality, which is essentially the factor responsible for the ellipticity, can fail for bounded, measurable coefficient satisfying the algebraic condition formally (see [55], [74] etc for such counterexamples in slightly different, but intimately related settings).*

Proof We shall only show the theorem under the assumption of Legendre-Hadamard condition on A , the other case being similar and easier. We will proceed in three steps.

Step 1 First assume $A(x) = \text{constant}$ and $B = C = 0$.

Since $C_c^\infty(\Omega; \Lambda^k)$ are dense in $W_T^{d,2}(\Omega, \Lambda^k)$, it is enough to show the inequality for $u \in C_c^\infty(\Omega; \Lambda^k)$. Now we have, using Fourier transform and Parseval-Plancherel identity and the hypothesis on A ,

$$\begin{aligned} \int_{\Omega} \langle Adu, du \rangle &= \int_{\mathbb{R}^n} \langle Adu, du \rangle = \int_{\mathbb{R}^n} \langle A \xi \wedge \hat{u}, \xi \wedge \hat{u} \rangle \\ &\geq \gamma \int_{\mathbb{R}^n} \langle \xi \wedge \hat{u}, \xi \wedge \hat{u} \rangle = \gamma \int_{\mathbb{R}^n} \langle du, du \rangle = \gamma \int_{\Omega} \langle du, du \rangle = \gamma \|du\|_{L^2}^2. \end{aligned}$$

Step 2 We now remove the hypothesis that A has constant coefficients but assume that support of u is small. We still keep the assumption that $B = C = 0$. By uniform continuity of A , there exists $\delta > 0$ such that,

$$|A(x) - A(y)| \leq \frac{\gamma}{2} \quad \text{whenever } |x - y| < \delta.$$

We now claim that for any $u \in C_c^\infty(\Omega; \Lambda^k)$ with $\text{diam}(\text{supp } u) < \delta$, we have,

$$\int_{\Omega} \langle A(x) du, du \rangle \geq \frac{\gamma}{2} \int_{\Omega} |du|^2. \quad (6.4)$$

To show this, we choose a point x_0 in the support of u and write,

$$\begin{aligned} \int_{\Omega} \langle A(x) du, du \rangle &= \int_{\Omega} \langle A(x_0) du, du \rangle + \int_{\text{supp } u} \langle (A(x) - A(x_0)) du, du \rangle \\ &\geq \gamma \int_{\Omega} |du|^2 - \frac{\gamma}{2} \int_{\Omega} |du|^2 \geq \frac{\gamma}{2} \int_{\Omega} |du|^2. \end{aligned}$$

Step 3 Now we finally remove the hypotheses that $B = C = 0$ and support of u is small. We now cover Ω with finitely many open balls $\{B_{\frac{\delta}{4}}(x_i)\}$ with $x_i \in \Omega$, $i = 1, 2, \dots, N$. We are now going to construct a special type of partition of unity for this cover. To this end, let $\zeta_i \in C_c^\infty(B_{\frac{\delta}{2}}(x_i))$ such that $\zeta_i(x) = 1$ for all $x \in B_{\frac{\delta}{4}}(x_i)$. We define,

$$\phi_i(x) = \frac{\zeta_i(x)}{\left(\sum_{j=1}^N \zeta_j(x)\right)^{\frac{1}{2}}} \quad \text{for } i = 1, 2, \dots, N.$$

Then we have $\sum_{i=1}^N \phi_i^2(x) = 1$ for all $x \in \Omega$. Then we have,

$$\int_{\Omega} \langle A(x) du, du \rangle = \sum_{i=1}^N \int_{\Omega} \langle A(x) (\phi_i^2(x) du), du \rangle = \sum_{i=1}^N \int_{\Omega} \langle A(x) (\phi_i(x) du), \phi_i(x) du \rangle.$$

Since,

$$\begin{aligned} \langle A(x) d(\phi_i(x) u), d(\phi_i(x) u) \rangle &= \langle A(x) (d\phi_i \wedge u), d\phi_i \wedge u \rangle + \langle A(x) (d\phi_i \wedge u), \phi_i u \rangle \\ &\quad + \langle A(x) (\phi_i du), d\phi_i \wedge u \rangle + \langle A(x) (\phi_i du), \phi_i du \rangle, \end{aligned}$$

we have,

$$\begin{aligned} &\sum_{i=1}^N \int_{\Omega} \langle A(x) (\phi_i(x) du), \phi_i(x) du \rangle \\ &= \sum_{i=1}^N \int_{\Omega} \langle A(x) d(\phi_i(x) u), d(\phi_i(x) u) \rangle - \sum_{i=1}^N \int_{\Omega} \langle A(x) (d\phi_i \wedge u), d\phi_i \wedge u \rangle \\ &\quad - \sum_{i=1}^N \int_{\Omega} \langle A(x) (d\phi_i \wedge u), \phi_i u \rangle - \sum_{i=1}^N \int_{\Omega} \langle A(x) (\phi_i du), d\phi_i \wedge u \rangle. \end{aligned}$$

Since $\phi_i u \in C_c^\infty(\Omega; \Lambda^k)$ with $\text{diam}(\text{supp } \phi_i u) \leq \text{diam}(B_{\frac{\delta}{2}}(x_i)) \leq \frac{\delta}{2}$, we have, by Step 2,

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} \langle A(x) d(\phi_i(x)u), d(\phi_i(x)u) \rangle \\
& \geq \frac{\gamma}{2} \sum_{i=1}^N \int_{\Omega} |d(\phi_i u)|^2 \\
& = \frac{\gamma}{2} \sum_{i=1}^N \int_{\Omega} \phi_i^2 |du|^2 + \frac{\gamma}{2} \sum_{i=1}^N \int_{\Omega} |d\phi_i \wedge u|^2 + \gamma \sum_{i=1}^N \int_{\Omega} \langle \phi_i du, d\phi_i \wedge u \rangle \\
& = \frac{\gamma}{2} \int_{\Omega} |du|^2 + \frac{\gamma}{2} \sum_{i=1}^N \int_{\Omega} |d\phi_i \wedge u|^2 + \gamma \sum_{i=1}^N \int_{\Omega} \langle \phi_i du, d\phi_i \wedge u \rangle.
\end{aligned}$$

Now we also have the following estimates, where c denotes a generic positive constant depending on ϕ_i and L^∞ norms of A, B, C , which may not represent the same constant in each line,

$$\begin{aligned}
& - \sum_{i=1}^N \int_{\Omega} \langle A(x) (d\phi_i \wedge u), d\phi_i \wedge u \rangle \geq -c \|u\|_{L^2}^2, \\
& - \sum_{i=1}^N \int_{\Omega} \langle A(x) (d\phi_i \wedge u), \phi_i u \rangle \geq -c \|u\|_{L^2}^2, \\
& - \sum_{i=1}^N \int_{\Omega} \langle A(x) (\phi_i du), d\phi_i \wedge u \rangle \geq -c \|u\|_{L^2} \|du\|_{L^2}, \\
& \frac{\gamma}{2} \sum_{i=1}^N \int_{\Omega} |d\phi_i \wedge u|^2 \geq -c \|u\|_{L^2}^2, \\
& \gamma \sum_{i=1}^N \int_{\Omega} \langle \phi_i du, d\phi_i \wedge u \rangle \geq -c \|u\|_{L^2} \|du\|_{L^2}, \\
& \int_{\Omega} \langle B(x) u, du \rangle \geq -c \|u\|_{L^2} \|du\|_{L^2}, \\
& \int_{\Omega} \langle C(x) u, u \rangle \geq -c \|u\|_{L^2}^2
\end{aligned}$$

Combining all the above estimates we deduce,

$$\begin{aligned}
& \int_{\Omega} [\langle A(x) du, du \rangle + \langle B(x) u, du \rangle + \langle C(x) du, u \rangle + \langle D(x) u, u \rangle] \\
& \geq \frac{\gamma}{2} \int_{\Omega} |du|^2 - C_1 \|u\|_{L^2} \|du\|_{L^2} - C_2 \|u\|_{L^2}^2.
\end{aligned}$$

Using Young's inequality with ε , we obtain,

$$-C_1 \|u\|_{L^2} \|du\|_{L^2} \geq -\varepsilon C_1 \|du\|_{L^2}^2 - \frac{1}{\varepsilon} C_1 \|u\|_{L^2}^2.$$

Choosing ε such that $\lambda_0 = \frac{\gamma}{2} - \varepsilon C_1 > 0$ and setting $\lambda_1 = \frac{1}{\varepsilon} C_1 + C_2$ for such a choice of ε , we obtain,

$$a(u, u) \geq \lambda_0 \|du\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2.$$

This completes the proof. ■

Remark 6.5 (i) the constant λ_1 can be chosen to be nonnegative, if one so desires. Since if $\lambda_1 < 0$, then $a(u, u) \geq \lambda_0 \|du\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2 \geq \lambda_0 \|du\|_{L^2}^2$.

(ii) As step 1 of the proof shows, if $A(x) = \text{constant}$ and satisfies Legendre-Hadamard and $B, C, D = 0$, then the inequality holds with $\lambda_1 = 0$. Also, if $B, C, D = 0$, then $\lambda_1 = 0$ for any A satisfying Legendre ellipticity.

Now we are ready to deduce existence of solutions in $W_{\delta, T}^{d, 2}(\Omega; \Lambda^k)$. We start with a few propositions.

Proposition 6.6 Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfy either the Legendre-Hadamard condition and is uniformly continuous or the Legendre condition and is bounded and measurable. Let $B \in L^\infty(\Omega; L(\Lambda^k, \Lambda^{k+1}))$, $C \in L^\infty(\Omega; L(\Lambda^{k+1}, \Lambda^k))$ and $D \in L^\infty(\Omega; L(\Lambda^k, \Lambda^k))$. Then for any $f \in L^2(\Omega, \Lambda^k)$ and $F \in L^2(\Omega, \Lambda^{k+1})$, there exists a constant $\tilde{\lambda}$ such that for any constant $\lambda \geq \tilde{\lambda}$, there exists unique $\omega \in W_{\delta, T}^{d, 2}(\Omega; \Lambda^k)$ satisfying,

$$\begin{aligned} \int_{\Omega} [\langle A(x)d\omega, d\theta \rangle + \langle B(x)\omega, d\theta \rangle + \langle C(x)d\omega, \theta \rangle + \langle D(x)\omega, \theta \rangle] + \lambda \int_{\Omega} \langle \omega, \theta \rangle \\ + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0, \end{aligned}$$

for all $\theta \in W_{\delta, T}^{d, 2}(\Omega; \Lambda^k)$.

Proof The plan is to use Lax-Milgram theorem. We recall that the norm $\|v\|_{W_{\delta, T}^{d, 2}(\Omega; \Lambda^k)} = \|dv\|_{L^2}$ is an equivalent norm on $W_{\delta, T}^{d, 2}(\Omega; \Lambda^k)$. For a given $\lambda \in \mathbb{R}$, we define the bilinear operators $a : W_{\delta, T}^{d, 2}(\Omega; \Lambda^k) \times W_{\delta, T}^{d, 2}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ and $b_\lambda : W_{\delta, T}^{d, 2}(\Omega; \Lambda^k) \times W_{\delta, T}^{d, 2}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ by,

$$\begin{aligned} a(u, v) &= \int_{\Omega} [\langle A(x)du, dv \rangle + \langle B(x)u, dv \rangle + \langle C(x)du, v \rangle + \langle D(x)u, v \rangle], \\ b_\lambda(u, v) &= a(u, v) + \lambda \int_{\Omega} \langle u, v \rangle. \end{aligned}$$

Clearly, $a(u, v)$ is continuous and so is $b_\lambda(u, v)$ for any $\lambda \in \mathbb{R}$, so we need only check the coercivity. Since $W_{\delta, T}^{d, 2}(\Omega; \Lambda^k) \subset W_T^{d, 2}(\Omega, \Lambda^k)$, by theorem 6.3, there exists constants $\lambda_0 > 0$ and λ_1 such that,

$$a(v, v) \geq \lambda_0 \|dv\|_{L^2}^2 - \lambda_1 \|v\|_{L^2}^2,$$

Set $\tilde{\lambda} = \lambda_1$. Then for any $\lambda \geq \tilde{\lambda}$, we have,

$$\begin{aligned} b_\lambda(v, v) &= a(v, v) + \lambda \int_{\Omega} \langle v, v \rangle = a(v, v) + \lambda \|v\|_{L^2}^2 \geq \lambda_0 \|dv\|_{L^2}^2 - \lambda_1 \|v\|_{L^2}^2 + \lambda \|v\|_{L^2}^2 \\ &= \lambda_0 \|dv\|_{L^2}^2 + (\lambda - \lambda_1) \|v\|_{L^2}^2 \geq \lambda_0 \|dv\|_{L^2}^2. \end{aligned}$$

Since $\lambda_0 > 0$, this shows coercivity and by Lax-Milgram theorem implies the existence of $\omega \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ satisfying

$$b_\lambda(\omega, \theta) = - \int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle \quad \text{for all } \theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k).$$

This completes the proof. ■

Remark 6.7 *This proposition above remains true even if the space $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is replaced by the larger space $W_T^{d,2}(\Omega, \Lambda^k)$. The only change in the proof is that we need to take the lower bound for λ , i.e $\tilde{\lambda} > \lambda_1$, where λ_1 is the constant in theorem 6.3, so that we can obtain, for any $\lambda \geq \tilde{\lambda}$,*

$$b_\lambda(v, v) \geq c \|v\|_{W^{d,2}}^2 \quad \text{with } c > 0.$$

The proposition above furnishes us with a ‘solution operator’. If $\bar{\lambda} \geq \tilde{\lambda}$, then the operator $T_{\bar{\lambda}} : (W_{\delta,T}^{d,2}(\Omega; \Lambda^k))^* \rightarrow W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, which maps the functional $\mathcal{F} \in (W_{\delta,T}^{d,2}(\Omega; \Lambda^k))^*$, given by,

$$\mathcal{F}(\theta) := - \int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle \quad \text{for all } \theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k),$$

for given $(f, F) \in L^2(\Omega, \Lambda^k) \times L^2(\Omega, \Lambda^{k+1})$ to the ‘solution’ α , i.e $\alpha \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is the unique solution to the problem,

$$b_{\bar{\lambda}}(\alpha, \theta) = - \int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle \quad \text{for all } \theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k),$$

is a bounded linear operator. Also, let $\mathcal{I} : W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \rightarrow (W_{\delta,T}^{d,2}(\Omega; \Lambda^k))^*$ be the embedding defined by,

$$\mathcal{I}v(\theta) = \int_{\Omega} \langle v, \theta \rangle. \quad (6.5)$$

We start with a lemma.

Lemma 6.8 *The operator $K_{\bar{\lambda}} : W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \rightarrow W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, given by $K_{\bar{\lambda}} = T_{\bar{\lambda}} \circ \mathcal{I}$ is a compact operator.*

Proof Since $T_{\bar{\lambda}}$ is continuous, it is enough to prove that \mathcal{I} is compact. But we can write $\mathcal{I} = \mathcal{I}_1 \circ \mathcal{I}_2$, where $\mathcal{I}_2 : W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \rightarrow L^2(\Omega, \Lambda^k)$ is the natural embedding and $\mathcal{I}_1 : L^2(\Omega, \Lambda^k) \rightarrow (W_{\delta,T}^{d,2}(\Omega; \Lambda^k))^*$ is given by (6.5). Since $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ compactly embeds in $L^2(\Omega, \Lambda^k)$, \mathcal{I}_2 is compact. Continuity of \mathcal{I}_1 concludes the proof. ■

Remark 6.9 *Note that since $W_T^{d,2}(\Omega, \Lambda^k)$ does not embed compactly in $L^2(\Omega, \Lambda^k)$, this lemma fails if $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is replaced by the larger space $W_T^{d,2}(\Omega, \Lambda^k)$.*

Theorem 6.10 *Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfy either the Legendre-Hadamard condition and is uniformly continuous or the Legendre condition and is bounded and measurable. Also let $B \in L^\infty(\Omega; L(\Lambda^k, \Lambda^{k+1}))$, $C \in L^\infty(\Omega; L(\Lambda^{k+1}, \Lambda^k))$ and $D \in L^\infty(\Omega; L(\Lambda^k, \Lambda^k))$. Then there exists*

a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$ such that the integro-differential equation,

$$\begin{aligned} \int_{\Omega} [\langle A(x)d\omega, d\theta \rangle + \langle B(x)\omega, d\theta \rangle + \langle C(x)d\omega, \theta \rangle + \langle D(x)\omega, \theta \rangle] + \lambda \int_{\Omega} \langle \omega, \theta \rangle \\ + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0, \end{aligned} \quad (6.6)$$

for all $\theta \in W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$, has a unique solution $\omega \in W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$ for all $f \in L^2(\Omega, \Lambda^k)$, $F \in L^2(\Omega, \Lambda^{k+1})$ if and only if $\lambda \notin \sigma$. Moreover, the set σ does not have a limit point except possibly $-\infty$. If σ is infinite, then it is a non-increasing sequence $\{\lambda_i\}$ such that $\lambda_i \rightarrow -\infty$ as $i \rightarrow \infty$. Also, for every $\sigma_i \in \sigma$, there exists non-trivial solutions $\alpha \in W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$, $\alpha \neq 0$ which solves the following integro-differential equation,

$$\int_{\Omega} [\langle A(x)d\omega, d\theta \rangle + \langle B(x)\omega, d\theta \rangle + \langle C(x)d\omega, \theta \rangle + \langle D(x)\omega, \theta \rangle] + \sigma_i \int_{\Omega} \langle \omega, \theta \rangle = 0$$

for all $\theta \in W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$. Moreover, the subspace of such solutions is finite dimensional.

Proof Let $g \in (W_{\delta, T}^{d,2}(\Omega; \Lambda^k))^*$ be given by,

$$g(\theta) = \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle \quad \text{for all } \theta \in W_{\delta, T}^{d,2}(\Omega; \Lambda^k).$$

A simple calculation shows that solving (6.6) is equivalent to solving the following functional equation on $W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$,

$$[I - (\bar{\lambda} - \lambda)K_{\bar{\lambda}}]\omega = T_{\bar{\lambda}}(g), \quad (6.7)$$

where $\bar{\lambda}, K_{\bar{\lambda}}, T_{\bar{\lambda}}$ are as defined above with $\bar{\lambda} \geq \tilde{\lambda}$, where $\tilde{\lambda}$ is the constant given by proposition 6.6 and $I : W_{\delta, T}^{d,2}(\Omega; \Lambda^k) \rightarrow W_{\delta, T}^{d,2}(\Omega; \Lambda^k)$ is the identity operator.

Now by lemma 6.8 $K_{\bar{\lambda}}$ is a compact operator, hence by Fredholm alternative theorem (cf. Theorem 5.3 and 5.5 in [34]) the theorem follows. Note that Fredholm alternative theorem yields the only possible limit point for the sequence $\left\{ \frac{1}{(\bar{\lambda} - \lambda_i)} \right\}_{\lambda_i \in \sigma}$ is 0. Since we already know that we can solve (6.6) uniquely for all $\lambda > \tilde{\lambda}$, we immediately deduce that the only possible limit point for $\sigma = \{\lambda_i\}_{i=1}^{\infty}$ must be $-\infty$ and by setting $\rho = \tilde{\lambda}$, $\sigma \subset (-\infty, \rho)$. Clearly, the set σ can be arranged in a non-increasing manner. ■

6.2.2 Existence in $W_T^{d,2}$

We shall now be interested in a solution of the integro-differential equation (6.6) on the larger space $W_T^{d,2}(\Omega, \Lambda^k)$, i.e we want to solve,

$$\begin{aligned} \int_{\Omega} [\langle A(x)d\omega, d\phi \rangle + \langle B(x)\omega, d\phi \rangle + \langle C(x)d\omega, \phi \rangle + \langle D(x)\omega, \phi \rangle] + \lambda \int_{\Omega} \langle \omega, \phi \rangle \\ + \int_{\Omega} \langle f, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0 \quad (\circ) \end{aligned}$$

for all $\phi \in W_T^{d,2}(\Omega, \Lambda^k)$. By proposition 6.6 and remark 6.7, we can always solve (o) if λ is large enough. However, as we already mentioned in remark 6.9, since the lemma 6.8 is no longer true, we can not use Fredholm alternative to infer about the solvability of (o) for any $\lambda \in \mathbb{R}$. In short, the lower order terms, in general, can not be treated as compact perturbations of the principal order term on $W_T^{d,2}(\Omega, \Lambda^k)$.

However if we assume additional conditions, it is possible to deduce some results.

6.2.3 Existence theorems

We are going to assume that the maps $B, C, D = 0$ and f is coclosed in the sense of distributions. Since our domain Ω is assumed contractible, any coclosed form is actually also coexact and hence we shall henceforth assume also $f = 0$. Under this assumption, it is possible to deduce existence and spectral theory not only on $W_T^{d,2}(\Omega, \Lambda^k)$ but actually in $W_T^{1,2}(\Omega, \Lambda^k)$. Moreover, we can derive the existence of a solution of the integro-differential equation and also for a related integro-differential equation on $W_T^{1,2}$, which will be crucially important to deduce regularity. This is the content of the following theorem.

Theorem 6.11 (Existence of weak solutions) *Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set. Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfy either the Legendre-Hadamard condition and is uniformly continuous or the Legendre condition and is L^∞ . Also let $F \in L^2(\Omega, \Lambda^{k+1})$. Then there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$, such that if $\lambda \notin \sigma$, then there exists a unique weak solution $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,*

$$\begin{cases} \delta(A(x)d\omega) = \lambda\omega + \delta F \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{P}_0)$$

Moreover $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ also satisfies all of the following integro-differential equations,

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0 \quad \text{for all } \phi \in W_T^{d,2}(\Omega, \Lambda^k). \quad (6.8)$$

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0 \quad \text{for all } \phi \in W_T^{1,2}(\Omega, \Lambda^k). \quad (6.9)$$

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0 \quad \text{for all } \phi \in W_T^{1,2}(\Omega, \Lambda^k). \quad (6.10)$$

Also for each $\sigma_i \in \sigma$ there exists non-trivial weak solutions $\alpha \in W_T^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\alpha) = \sigma_i\alpha \text{ in } \Omega, \\ \delta\alpha = 0 \text{ in } \Omega, \\ \nu \wedge \alpha = 0 \text{ on } \partial\Omega, \end{cases} \quad (\text{EV})$$

and the space of weak solutions to (EV) is finite-dimensional for any $\sigma_i \in \sigma$.

Remark 6.12 (i) Note that ω given by the preceding theorem is the unique solution to the boundary value problem (\mathcal{P}_0) , but it is not necessarily the unique solution to the integro-differential equations (6.8), (6.9), (6.10). This would in general require additional hypotheses. As a particular example of this non-uniqueness, if $0 \notin \sigma$, then setting $\lambda = 0$, we see that if ω solves (6.8) or (6.9), so does $\omega + d\psi$ for any $\psi \in W_0^{1,2}(\Omega; \Lambda^k)$.

(ii) For much of the same reason, in the preceding theorem, the space of weak solutions of the problem (EV) is finite dimensional for any $\sigma_i \in \sigma$, but the space of weak solutions to the problem,

$$\begin{cases} \delta(A(x)d\alpha) = \sigma_i \alpha \text{ in } \Omega, \\ \nu \wedge \alpha = 0 \text{ on } \partial\Omega, \end{cases} \quad (\text{EVP})$$

when $\sigma_i \in \sigma$ need not be finite dimensional. If $\sigma_i = 0 \in \sigma$, the space of weak solutions corresponding to (EV) would be finite dimensional, but the space of weak solutions to (EVP) to is clearly infinite-dimensional, as it contains $dW_0^{1,2}(\Omega; \Lambda^k)$.

Remark 6.13 Note that if $A \in L^\infty(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ is symmetric and satisfies the Legendre condition, using techniques similar to theorem 3.69, we can deduce that there exists a minimizer of the following problem,

$$m = \inf \left\{ \int_{\Omega} [\langle A(x)d\omega, d\omega \rangle + \lambda|\omega|^2 - \langle F, d\omega \rangle] : \omega \in \omega_0 + W_T^{d,2}(\Omega; \Lambda^k) \right\},$$

for any $F \in L^2(\Omega; \Lambda^{k+1})$, for any $\omega_0 \in W^{d,2}(\Omega; \Lambda^k)$ when $\lambda > 0$ is large enough. However, the minimizer is only in $W_T^{d,2}(\Omega; \Lambda^k)$, whereas theorem 6.11 gives a solution in $W_T^{1,2}(\Omega; \Lambda^k)$. This additional gain in regularity is significant.

Proof We prove only the case of Legendre-Hadamard ellipticity, the other case is handled exactly similarly.

The hypothesis of the theorem implies, by theorem 6.10, that there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$ such that the integro-differential equation,

$$\int_{\Omega} \langle A(x)d\omega, d\theta \rangle + \lambda \int_{\Omega} \langle \omega, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0 \quad \text{for all } \theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \quad (6.11)$$

has a unique solution $\omega \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ for any $F \in L^2(\Omega; \Lambda^{k+1})$ if and only if $\lambda \notin \sigma$. Moreover, the set σ does not have a limit point except possibly $-\infty$. If σ is infinite, then it is a non-increasing sequence $\{\lambda_i\}$ such that $\lambda_i \rightarrow -\infty$ as $i \rightarrow \infty$. Also, for every $\sigma_i \in \sigma$, there exists a finite dimensional subspace of solutions, containing non-trivial solutions $\alpha \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, $\alpha \neq 0$ solving the following integro-differential equation,

$$\int_{\Omega} \langle A(x)d\alpha, d\theta \rangle + \sigma_i \int_{\Omega} \langle \alpha, \theta \rangle = 0 \quad \text{for all } \theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k). \quad (6.12)$$

We first tackle the last half of the theorem. We recall that $W_T^{1,2}(\Omega; \Lambda^k)$ is a subspace of $W_T^{d,2}(\Omega; \Lambda^k)$ and the orthogonal decomposition $W_T^{d,2}(\Omega; \Lambda^k) = W_{\delta,T}^{d,2}(\Omega; \Lambda^k) \oplus dW_0^{1,2}(\Omega; \Lambda^k)$.

Hence we can write every $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$ as $\phi = \theta + d\psi$, for some $\theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, $\psi \in W_0^{1,2}(\Omega; \Lambda^k)$. Now if $\sigma_i \in \sigma$ and $\alpha \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is a non-trivial solution of (6.12), then we have, for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$,

$$\begin{aligned} \int_{\Omega} \langle A(x)d\alpha, d\phi \rangle + \sigma_i \int_{\Omega} \langle \alpha, \phi \rangle &= \int_{\Omega} \langle A(x)d\alpha, d(\theta + d\psi) \rangle + \sigma_i \int_{\Omega} \langle \alpha, \theta + d\psi \rangle \\ &= \int_{\Omega} \langle A(x)d\alpha, d\theta \rangle + \sigma_i \int_{\Omega} \langle \alpha, \theta \rangle + \sigma_i \int_{\Omega} \langle \alpha, d\psi \rangle \\ &= 0, \end{aligned}$$

where the last term on the left of the last equality is 0 since $\alpha \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ and hence $\delta\alpha = 0$ in the sense of distributions and the rest is 0 by (6.12). Also, since $\alpha \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, clearly $\alpha \in W_T^{1,2}(\Omega; \Lambda^k)$, $\nu \wedge \alpha = 0$ on $\partial\Omega$ and $\delta\alpha = 0$ in Ω , showing that such an α is indeed a weak solution to (EV). This settles the last part of the theorem.

For the other part, for any $\lambda \notin \sigma$, if $\omega \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$ is the unique solution of (6.11), then since we can write any $\phi \in W_T^{d,2}(\Omega; \Lambda^k)$ as $\phi = \theta + d\psi$, for some $\theta \in W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$, $\psi \in W_0^{1,2}(\Omega; \Lambda^k)$, we deduce, for all $\phi \in W_T^{d,2}(\Omega; \Lambda^k)$,

$$\begin{aligned} \int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle \\ &= \int_{\Omega} \langle A(x)d\omega, d(\theta + d\psi) \rangle + \lambda \int_{\Omega} \langle \omega, \theta + d\psi \rangle - \int_{\Omega} \langle F, d(\theta + d\psi) \rangle \\ &= \int_{\Omega} \langle A(x)d\omega, d\theta \rangle + \lambda \int_{\Omega} \langle \omega, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle + \lambda \int_{\Omega} \langle \omega, d\psi \rangle \\ &= 0, \end{aligned}$$

where the last term on the left of the last equality vanishes since $\delta\omega = 0$ in the sense of distributions and the rest is 0 by (6.11). This proves that ω solves (6.8). Since $W_T^{1,2}(\Omega; \Lambda^k)$ is a subspace of $W_T^{d,2}(\Omega; \Lambda^k)$, this immediately implies ω solves (6.9). Clearly (6.9) implies that ω is a weak solution to the boundary value problem (\mathcal{P}_0) . Since $\delta\omega$ must be 0 for any solution of (\mathcal{P}_0) , uniqueness follows from uniqueness of ω in $W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$. Again, since $\delta\omega = 0$ in the sense of distributions, we have,

$$\int_{\Omega} \langle \delta\omega, \delta\phi \rangle = 0 \quad \text{for all } \phi \in W_T^{1,2}(\Omega; \Lambda^k).$$

This together with (6.9) implies ω solves (6.10). This completes the proof. ■

6.3 Interior regularity of weak solutions

We now prove the interior regularity results. We deduce the interior regularity results for linear Maxwell operator from the classical interior regularity results for a linear elliptic system. The point is that for interior regularity results, the boundary conditions do not matter and hence deducing interior regularity results follow from the classical ones as soon as we show that the system we are dealing with is in fact elliptic. We start with the following theorem.

Theorem 6.14 (Interior $W^{2,2}$ regularity) *Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ be Lipschitz continuous and satisfies either the Legendre-Hadamard or Legendre condition. Also let $f \in L^2(\Omega, \Lambda^k)$ and $\lambda \in \mathbb{R}$. Let $\omega \in W^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_{\Omega} \langle A(x) d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0 \quad \text{for all } \phi \in W_T^{1,2}(\Omega, \Lambda^k). \quad (6.13)$$

Then $\omega \in W_{loc}^{2,2}(\Omega, \Lambda^k)$, and for any subdomain $\Omega' \subset\subset \Omega$, there is a constant C , depending only on Ω, Ω' and Lipschitz norm of A , such that we have the estimate,

$$\|\omega\|_{W^{2,2}(\Omega'; \Lambda^k)} \leq C \left(\|\omega\|_{L^2(\Omega; \Lambda^k)} + \|f\|_{L^2(\Omega; \Lambda^k)} \right).$$

To show this, we shall first need to show that the system we are dealing with is in fact elliptic. This is the content of the following lemma.

Lemma 6.15 (ellipticity lemma) *Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ be a measurable map and satisfies,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma > 0$ for all $x \in \Omega$. We define the map $\tilde{A} : \Omega \rightarrow L(\mathbb{R}^{\binom{n}{k} \times n}, \mathbb{R}^{\binom{n}{k} \times n})$ by,

$$\tilde{A}(x) = (\pi^{ext, k+1})^T \circ A(x) \circ \pi^{ext, k+1} + (\pi^{int, k-1})^T \circ \pi^{int, k-1} \quad \text{for a.e } x \in \Omega,$$

where $\pi^{ext, k+1}, \pi^{int, k-1}$ are the projection maps defined in chapter 3 and $(\cdot)^T$ denotes the transpose. Then \tilde{A} satisfies,

$$\langle \tilde{A}(x)(a \otimes b) ; a \otimes b \rangle \geq \gamma_0 |a|^2 |b|^2, \quad \text{for every } a \in \mathbb{R}^n, b \in \mathbb{R}^{\binom{n}{k}}$$

for some constant $\gamma_0 > 0$ for a.e $x \in \Omega$.

Remark 6.16 (1) *As usual, we identify Λ^1 with \mathbb{R}^n and Λ^k with $\mathbb{R}^{\binom{n}{k}}$.*

(2) *Observe that since $\pi^{ext, k+1}, \pi^{int, k-1}$ are linear maps with constant coefficients, \tilde{A} always enjoys the same regularity as A .*

(3) *The conclusion of the lemma shows that \tilde{A} satisfies the Legendre-Hadamard ellipticity condition or strong ellipticity condition in the sense of linear elliptic systems.*

(4) *The definition of \tilde{A} implies, for a.e $x \in \Omega$ and for every $a \in \Lambda^1, b \in \Lambda^k$,*

$$\langle \tilde{A}(x)(a \otimes b); a \otimes b \rangle = \langle A(x)(a \wedge b) ; a \wedge b \rangle + \langle a \lrcorner b; a \lrcorner b \rangle.$$

We shall show this while proving the lemma.

(5) *In the same manner, we have, for a.e $x \in \Omega$ and for every $\omega, \phi \in W^{d,2}(\Omega, \Lambda^k)$,*

$$\langle \tilde{A}(x)(\nabla\omega); \nabla\phi \rangle = \langle A(x)d\omega; d\phi \rangle + \langle \delta\omega; \delta\phi \rangle.$$

This observation is the crucial one by virtue of which we can deduce all the regularity results from the classical results.

(6) Note however that, if A satisfies the Legendre condition, i.e if there is a constant $\gamma > 0$ such that,

$$\langle A(x)\lambda ; \lambda \rangle \geq \gamma |\lambda|^2, \quad \text{for every } \lambda \in \Lambda^{k+1}, \text{ for a.e } x \in \Omega,$$

this still would not imply that there is a constant $\gamma_0 > 0$ such that,

$$\langle \tilde{A}(x)\xi ; \xi \rangle \geq \gamma_0 |\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^{\binom{n}{k} \times n}, \text{ for a.e } x \in \Omega.$$

The conclusion of the lemma would still hold though, since Legendre condition on A implies the Legendre-Hadamard condition for \tilde{A} .

Proof For any $a \in \Lambda^1, b \in \Lambda^k$, we have, by abuse of notations,

$$\begin{aligned} & \langle \tilde{A}(x)(a \otimes b); a \otimes b \rangle \\ &= \langle [(\pi^{ext,k+1})^T \circ A(x) \circ \pi^{ext,k+1} + (\pi^{int,k-1})^T \circ \pi^{int,k-1}] (a \otimes b); a \otimes b \rangle \\ &= \langle [(\pi^{ext,k+1})^T \circ A(x) \circ \pi^{ext,k+1}] (a \otimes b); a \otimes b \rangle + \langle [(\pi^{int,k-1})^T \circ \pi^{int,k-1}] (a \otimes b); a \otimes b \rangle \\ &= \langle (A(x) \circ \pi^{ext,k+1})(a \otimes b); \pi^{ext,k+1}(a \otimes b) \rangle + \langle \pi^{int,k-1}(a \otimes b); \pi^{int,k-1}(a \otimes b) \rangle \\ &= \langle A(x)(\pi^{ext,k+1}(a \otimes b)); \pi^{ext,k+1}(a \otimes b) \rangle + \langle \pi^{int,k-1}(a \otimes b); \pi^{int,k-1}(a \otimes b) \rangle \\ &= \langle A(x)(a \wedge b); a \wedge b \rangle + \langle a \lrcorner b; a \lrcorner b \rangle. \end{aligned}$$

But, using the hypothesis on A , this implies,

$$\langle \tilde{A}(x)(a \otimes b); a \otimes b \rangle \geq \gamma |a \wedge b|^2 + |a \lrcorner b|^2.$$

We now claim that this implies there exists a constant $\gamma_0 > 0$ such that,

$$\gamma |a \wedge b|^2 + |a \lrcorner b|^2 \geq \gamma_0 |a|^2 |b|^2.$$

Clearly the claim establishes the lemma, so all that remains is to prove the claim. But if the claim is false, then there exist sequences $\{a_n\}, \{b_n\}$ such that for every $n \in \mathbb{N}$, we have,

$$\gamma |a_n \wedge b_n|^2 + |a_n \lrcorner b_n|^2 < \frac{1}{n} \quad \text{with } |a_n| = |b_n| = 1.$$

But since $\{a_n\}, \{b_n\}$ are bounded sequences, passing to a subsequence if necessary, we can assume that,

$$a_n \rightarrow a \quad \text{and} \quad b_n \rightarrow b \quad \text{as } n \rightarrow \infty \text{ with } |a| = |b| = 1.$$

Then, passing to the limit as $n \rightarrow \infty$, we obtain,

$$\gamma |a \wedge b|^2 + |a \lrcorner b|^2 = 0,$$

which implies $a \wedge b = 0$ and $a \lrcorner b = 0$. Plugging this in the identity (cf. Proposition 2.16 in [21]), $a \wedge (a \lrcorner b) + a \lrcorner (a \wedge b) = |a|^2 b$, we obtain $b = 0$, which contradicts the fact that $|b| = 1$ and finishes the proof. ■

Incidentally, such a lemma holds true even in more general circumstances. The proof is completely analogous to the lemma 6.15 with obvious changes and is omitted.

Lemma 6.17 (general ellipticity lemma) *Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ be a measurable map and satisfies,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma_1 |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma_2 > 0$ for all $x \in \Omega$. Also let $B : \Omega \rightarrow L(\Lambda^{k-1}, \Lambda^{k-1})$ be a measurable map and satisfies,

$$\langle B(x)(a \lrcorner b) ; a \lrcorner b \rangle \geq \gamma_2 |a \lrcorner b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma_2 > 0$ for all $x \in \Omega$. We define the map $\tilde{A} : \Omega \rightarrow L(\mathbb{R}^{\binom{n}{k} \times n}, \mathbb{R}^{\binom{n}{k} \times n})$ by,

$$\tilde{A}(x) = (\pi^{ext, k+1})^T \circ A(x) \circ \pi^{ext, k+1} + (\pi^{int, k-1})^T \circ B(x) \circ \pi^{int, k-1} \quad \text{for a.e } x \in \Omega,$$

where $\pi^{ext, k+1}, \pi^{int, k-1}$ are the projection maps defined in chapter 2 and $(\cdot)^T$ denotes the transpose. Then \tilde{A} satisfies,

$$\langle \tilde{A}(x)(a \otimes b) ; a \otimes b \rangle \geq \gamma_0 |a|^2 |b|^2, \quad \text{for every } a \in \mathbb{R}^n, b \in \mathbb{R}^{\binom{n}{k}}$$

for some constant $\gamma_0 > 0$ for a.e $x \in \Omega$.

This lemma is enough to prove theorem 6.14. Let us show that this indeed is the case.

Proof (of theorem 6.14) We define the measurable map $\tilde{A} : \Omega \rightarrow L(\mathbb{R}^{\binom{n}{k} \times n}, \mathbb{R}^{\binom{n}{k} \times n})$ by,

$$\tilde{A}(x) = (\pi^{ext, k+1})^T \circ A(x) \circ \pi^{ext, k+1} + (\pi^{int, k-1})^T \circ B(x) \circ \pi^{int, k-1} \quad \text{for a.e } x \in \Omega.$$

Note that the hypothesis of the theorem implies that $\tilde{A} \in W^{1, \infty}(\Omega; L(\mathbb{R}^{\binom{n}{k} \times n}, \mathbb{R}^{\binom{n}{k} \times n}))$.

Now for all $\phi \in W_T^{1, 2}(\Omega, \Lambda^k)$, we have.

$$\begin{aligned} & \int_{\Omega} \langle \tilde{A}(x)(\nabla \omega), \nabla \phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle \\ &= \int_{\Omega} \langle A(x)(\pi^{ext, k+1}(\nabla \omega)), \pi^{ext, k+1}(\nabla \phi) \rangle + \int_{\Omega} \langle \pi^{int, k-1}(\nabla \omega), \pi^{int, k-1}(\nabla \phi) \rangle \\ & \quad + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle \\ &= \int_{\Omega} \langle A(x) d\omega, d\phi \rangle + \int_{\Omega} \langle \delta \omega, \delta \phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle \\ &= 0. \end{aligned}$$

Since $W_0^{1,2}(\Omega, \Lambda^k) \subset W_T^{1,2}(\Omega, \Lambda^k)$, we see that $\omega \in W^{1,2}(\Omega, \Lambda^k)$ satisfies,

$$\int_{\Omega} \langle \tilde{A}(x)(\nabla \omega), \nabla \phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0 \text{ for all } \phi \in W_0^{1,2}(\Omega, \Lambda^k). \quad (6.14)$$

Since by lemma 6.15 \tilde{A} satisfies the classical Legendre-Hadamard condition, the classical results (for example cf. Theorem 4.9 in [33]) immediately imply $\omega \in W_{loc}^{2,2}(\Omega, \Lambda^k)$. ■

In exactly the same way, we can deduce the higher interior regularity result from the classical results (cf. Theorem 4.11 in [33]). We state the theorem below and omit the proof.

Theorem 6.18 (Interior $W^{r+2,2}$ regularity) *Let $1 \leq k \leq n-1$, $r \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $A \in C^{r,1}(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard or Legendre ellipticity condition. Also let $f \in W^{r,2}(\Omega, \Lambda^k)$ and $\lambda \in \mathbb{R}$. Let $\omega \in W^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0, \quad (6.15)$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. Then $\omega \in W_{loc}^{r+2,2}(\Omega, \Lambda^k)$, and for any subdomain $\Omega' \subset\subset \Omega$, there is a constant C , depending only on Ω, Ω' and $C^{r,1}$ norm of A , such that we have the estimate,

$$\|\omega\|_{W^{r+2,2}(\Omega'; \Lambda^k)} \leq C \left(\|\omega\|_{L^2(\Omega; \Lambda^k)} + \|f\|_{W^{r,2}(\Omega; \Lambda^k)} \right).$$

The argument outlined at the end of the last subsection is also enough to derive the interior regularity results in Hölder and $W^{m,p}$ spaces from the classical ones for linear elliptic systems (cf. e.g Theorem 5.20 and Theorem 7.2 in [33] for Schauder and L^p estimates respectively). Here we record the results.

Theorem 6.19 (Interior $C^{r+2,\alpha}$ regularity) *Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $0 < \alpha < 1$ be a real number and Let $A \in C^{r+1,\alpha}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard or Legendre ellipticity condition. Also $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ and $\lambda \in \mathbb{R}$. Let $\omega \in W^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0, \quad (6.16)$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. Then $\omega \in C_{loc}^{r+2,\alpha}(\Omega, \Lambda^k)$, and for any subdomain $\Omega' \subset\subset \Omega$, there is a constant C , depending only on Ω, Ω' and $C^{r+1,\alpha}$ norm of A , such that we have the estimate,

$$\|\omega\|_{C^{r+2,\alpha}(\overline{\Omega'}; \Lambda^k)} \leq C \left\{ \|\omega\|_{C^{0,\alpha}(\overline{\Omega}; \Lambda^k)} + \|f\|_{C^{r,\alpha}(\overline{\Omega}; \Lambda^k)} \right\}.$$

Theorem 6.20 (Interior $W^{r+2,p}$ regularity) *Let $1 \leq k \leq n-1$, $r \geq 0$ be integers and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $1 < p < \infty$ be a real number and let $A \in C^{r+1}(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard or Legendre ellipticity condition. Also $f \in W^{r,p}(\Omega, \Lambda^k)$ and $\lambda \in \mathbb{R}$. Let $\omega \in W^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,*

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0, \quad (6.17)$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. Then $\omega \in W_{loc}^{r+2,p}(\Omega, \Lambda^k)$, and for any subdomain $\Omega' \subset\subset \Omega$, there is a constant C , depending only on Ω, Ω' and C^{r+1} norm of A , such that we have the estimate,

$$\|\omega\|_{W^{r+2,p}(\Omega'; \Lambda^k)} \leq C \left\{ \|\omega\|_{L^p(\Omega; \Lambda^k)} + \|f\|_{W^{r,p}(\Omega; \Lambda^k)} \right\}.$$

Remark 6.21 Note that the terms containing the L^2 , $C^{0,\alpha}$ and L^p norm of ω , on the right hand side of the estimates in theorem 6.18, theorem 6.19 and theorem 6.20 respectively, can not in general be dropped because of possible nonuniqueness. Indeed, even when A satisfies the Legendre condition or satisfies only Legendre-Hadamard but has constant coefficients and $\lambda = 0$, uniqueness of solution is true only modulo harmonic fields.

6.4 Regularity up to the boundary

However, for deducing regularity up to the boundary we need something more. The reason is the special nature of the boundary conditions. In general, regularity results up to the boundary is not standard in the classical literature for such boundary conditions. Hence we would have to prove it for ourselves. First we need a few lemmas. We begin by recalling our framework.

Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set. Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ be a measurable map that satisfies,

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma_0 |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma_0 > 0$ for all $x \in \Omega$. Also let $f \in L^2(\Omega; \Lambda^k)$, $F \in L^2(\Omega; \Lambda^{k+1})$ and $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0, \quad (6.18)$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$.

Now we derive the integral equation satisfied by ω in a neighbourhood of the boundary, multiplied by a local cut off.

Lemma 6.22 If $x_0 \in \partial\Omega$, W be a neighbourhood of x_0 in \mathbb{R}^n and $\theta \in C_c^\infty(W)$. Let $V = \Omega \cap W$. Assume $A \in C^{0,1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$. If $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ is a weak solution of (6.18), then $\theta\omega$ satisfies the following equation,

$$\int_V \langle A(x)d(\theta\omega), d\phi \rangle + \int_V \langle \delta(\theta\omega), \delta\phi \rangle + \int_V \langle \theta f - d\theta \lrcorner F, \phi \rangle - \int_V \langle \theta F, d\phi \rangle + \int_V \langle g, \phi \rangle - \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle = 0, \quad (6.19)$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$, where g is given by,

$$g = \lambda\theta\omega + \delta(A(x)(d\theta \wedge \omega)) + d\theta \lrcorner (A(x)(d\omega)) + d\theta \wedge \delta\omega. \quad (6.20)$$

Proof This lemma is just a straight forward calculation. We have, for any $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$,

$$\begin{aligned}
& \int_V \langle A(x)d(\theta\omega), d\phi \rangle + \int_V \langle \delta(\theta\omega), \delta\phi \rangle \\
&= \int_V \langle A(x)(d\theta \wedge \omega + \theta d\omega), d\phi \rangle + \int_V \langle (d\theta \lrcorner \omega + \theta \delta\omega), \delta\phi \rangle \\
&= \int_V \langle A(x)(d\theta \wedge \omega), d\phi \rangle + \int_V \langle A(x)(d\omega), \theta d\phi \rangle + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle + \int_V \langle \delta\omega, \theta \delta\phi \rangle \\
&= \int_V \langle A(x)(d\theta \wedge \omega), d\phi \rangle + \int_V \langle A(x)(d\omega), (d(\theta\phi) - d\theta \wedge \phi) \rangle + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle \\
&\quad + \int_V \langle \delta\omega, (\delta(\theta\phi) - d\theta \lrcorner \phi) \rangle \\
&= \int_V \langle A(x)(d\omega), d(\theta\phi) \rangle + \int_V \langle \delta\omega, \delta(\theta\phi) \rangle + \int_V \langle A(x)(d\theta \wedge \omega), d\phi \rangle + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle \\
&\quad - \int_V \langle A(x)(d\omega), d\theta \wedge \phi \rangle - \int_V \langle \delta\omega, d\theta \lrcorner \phi \rangle.
\end{aligned}$$

But since $\theta\phi$ can be taken as a test function in (6.18), we can substitute the first two terms and obtain,

$$\begin{aligned}
& \int_V \langle A(x)d(\theta\omega), d\phi \rangle + \int_V \langle \delta(\theta\omega), \delta\phi \rangle \\
&= -\lambda \int_V \langle \omega, (\theta\phi) \rangle - \int_V \langle f, (\theta\phi) \rangle + \int_V \langle F, d(\theta\phi) \rangle + \int_V \langle A(x)(d\theta \wedge \omega), d\phi \rangle \\
&\quad + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle - \int_V \langle A(x)(d\omega), d\theta \wedge \phi \rangle - \int_V \langle \delta\omega, d\theta \lrcorner \phi \rangle \\
&= -\lambda \int_V \langle \omega, (\theta\phi) \rangle - \int_V \langle f, (\theta\phi) \rangle + \int_V \langle F, \theta d\phi \rangle + \int_V \langle F, d\theta \wedge \phi \rangle + \int_V \langle A(x)(d\theta \wedge \omega), d\phi \rangle \\
&\quad + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle - \int_V \langle A(x)(d\omega), d\theta \wedge \phi \rangle - \int_V \langle \delta\omega, d\theta \lrcorner \phi \rangle \\
&= -\lambda \int_V \langle \theta\omega, \phi \rangle - \int_V \langle \theta f, \phi \rangle + \int_V \langle \theta F, d\phi \rangle + \int_V \langle d\theta \lrcorner F, \phi \rangle - \int_V \langle \delta(A(x)(d\theta \wedge \omega)), \phi \rangle \\
&\quad + \int_V \langle d\theta \lrcorner \omega, \delta\phi \rangle - \int_V \langle d\theta \lrcorner (A(x)(d\omega)), \phi \rangle - \int_V \langle d\theta \wedge \delta\omega, \phi \rangle.
\end{aligned}$$

This, after transposing proves the result. \blacksquare

Flattening the boundary Now we flatten the boundary and derive the equation satisfied by the pullback of $\theta\omega$ in half balls in the half space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$. Here we shall be a bit more precise about the smoothness of the boundary.

Let $B_{R_0}^+$ denote the half-ball centered around 0 in the half space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$, i.e

$$B_{R_0}^+ = \{x \in \mathbb{R}^n : |x| < R_0, x_n > 0\}.$$

Let Γ_{R_0} denote the flat part of the boundary of the half ball $B_{R_0}^+$, i.e

$$\Gamma_{R_0} = \{x \in \mathbb{R}^n : |x| \leq R_0, x_n = 0\},$$

and let C_{R_0} denote the curved part of the boundary of the half ball $B_{R_0}^+$, i.e

$$C_{R_0} = \{x \in \mathbb{R}^n : |x| = R_0, x_n \geq 0\}.$$

Also let us denote the space of Sobolev functions with vanishing tangential component on the flat part of the boundary by,

$$W_{T,flat}^{1,2}(B_{R_0}^+; \Lambda^k) := \{\psi \in W^{1,2}(B_{R_0}^+; \Lambda^k) : \psi_T = -e_n \wedge \psi = 0 \text{ on } \Gamma_{R_0}\}.$$

We also define,

$$W_{T,flat}^{r,2}(B_{R_0}^+; \Lambda^k) = W_{T,flat}^{1,2}(B_{R_0}^+; \Lambda^k) \cap W^{r,2}(B_{R_0}^+; \Lambda^k), \quad \text{for every } r \geq 1.$$

Now let $r \geq 0$ be an integer and $0 < \gamma < 1$. If $\partial\Omega$ is of class C^{r+2} (respectively, $C^{r+2,\gamma}$), then for every $x_0 \in \partial\Omega$, we know there exists a neighbourhood W of x_0 in \mathbb{R}^n such that there is an admissible boundary coordinate system $\Phi \in \text{Diff}^{r+2}(\overline{B_{R_0}}; \overline{W})$ (respectively, $\Phi \in \text{Diff}^{r+2,\gamma}(\overline{B_{R_0}}; \overline{W})$) for some $R_0 > 0$ such that $\Phi(0) = x_0$ and $\Phi(B_{R_0}^+) = \Omega \cap W$. We now derive the equation satisfied by $u = \Phi^*(\theta\omega)$ in a half ball centered around 0 in $B_{R_0}^+$.

Lemma 6.23 *Let $r \geq 0$ be an integer and $0 < \gamma < 1$. Also let $\partial\Omega$ is of class C^{r+2} , respectively $C^{r+2,\gamma}$. Let $x_0 \in \partial\Omega$, W be a neighbourhood of x_0 in \mathbb{R}^n . Let $\Phi \in \text{Diff}^{r+2}(\overline{B_{R_0}}; \overline{W})$, respectively $\Phi \in \text{Diff}^{r+2,\gamma}(\overline{B_{R_0}}; \overline{W})$, be an admissible boundary coordinate system, for some $R_0 > 0$, such that $\Phi(0) = x_0$ and $\Phi(B_{R_0}^+) = \Omega \cap W$. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\gamma}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, satisfy,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma_0 |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma_0 > 0$ for all $x \in \Omega$. Also let $f \in W^{r_0,2}(\Omega; \Lambda^k)$ and $F \in W^{r_1,2}(\Omega; \Lambda^{k+1})$ for some integers $r+1 \geq r_1 \geq r_0 \geq r$.

If $\omega \in W_T^{1,2}(\Omega, \Lambda^k) \cap W^{r+1,2}(\Omega, \Lambda^k)$ satisfy

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0, \quad (6.21)$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$, then for every given $\varepsilon > 0$, there exist $\theta \in C_c^\infty(W)$, $R > 0$, $\bar{A} \in L(\Lambda^{k+1}, \Lambda^{k+1})$, $\tilde{f} \in W^{r,2}(B_R^+; \Lambda^k)$ and functions $a_\alpha \in W^{r,2}(B_R^+)$, $b_\alpha^i \in W^{r+1,2}(B_R^+)$, $p_{\alpha\beta}, q_{\alpha\beta}^i, r_{\alpha\beta}^i \in C^r(\overline{B_R^+})$, respectively $C^{r,\gamma}(\overline{B_R^+})$, $s_{\alpha\beta}^{ij} \in C^{r+1}(\overline{B_R^+})$, respectively $C^{r+1,\gamma}(\overline{B_R^+})$, such that $u = \Phi^*(\theta\omega) \in W_{T,flat}^{r+1,2}(B_R^+; \Lambda^k)$ vanishes in a neighbourhood of the curved part of the bound-

ary of B_R^+ and satisfies, for all $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$,

$$\begin{aligned} \int_{B_R^+} \langle \bar{A}(du); d\psi \rangle + \int_{B_R^+} \langle \delta u; \delta \psi \rangle + \int_{B_R^+} \langle \tilde{f}; \psi \rangle - \int_{B_R^+} \langle \tilde{F}; d\psi \rangle + \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha \psi_\alpha \\ + \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} b_\alpha^i \frac{\partial \psi^\alpha}{\partial x_i} + \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} p_{\alpha\beta} u^\alpha \psi^\beta + \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left(q_{\alpha\beta}^i \frac{\partial \psi^\alpha}{\partial x_i} u^\beta + r_{\alpha\beta}^i \frac{\partial u^\alpha}{\partial x_i} \psi^\beta \right) \\ + \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} s_{\alpha\beta}^{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_j} = 0, \end{aligned} \quad (6.22)$$

where the functions $p_{\alpha\beta}, q_{\alpha\beta}^i, r_{\alpha\beta}^i, s_{\alpha\beta}^{ij}$ depend only on A and Φ and satisfies,

$$\|s_{\alpha\beta}^{ij}\|_{C^r(\overline{B_R^+})} \text{ (respectively } \|s_{\alpha\beta}^{ij}\|_{C^{r,\gamma}(\overline{B_R^+})}) \leq \varepsilon, \text{ for all } i, j = 1, \dots, n \text{ and for all } \alpha, \beta \in \mathcal{T}^k \quad (6.23)$$

and $a_\alpha \in W^{r,2}(B_R^+)$ and $b_\alpha^i \in W^{r+1,2}(B_R^+)$ depend on ω , A and Φ and satisfies,

$$\|a_\alpha\|_{W^{r,2}(B_R^+)}, \|b_\alpha^i\|_{W^{r+1,2}(B_R^+)} \leq c_0 \|\omega\|_{W^{r+1,2}(\Omega, \Lambda^k)}, \text{ for all } i = 1, \dots, n \text{ and for all } \alpha \in \mathcal{T}^k, \quad (6.24)$$

where $c_0 > 0$ is a constant, depending only on Φ and A . Moreover, \bar{A} satisfies the Legendre-Hadamard condition, i.e there exists a constant $\tilde{\gamma}_0 > 0$ such that,

$$\langle \bar{A}(a \wedge b); a \wedge b \rangle \geq \tilde{\gamma}_0 |a \wedge b|^2 \quad \text{for all } a \in \Lambda^1, b \in \Lambda^k, \quad (6.25)$$

and $\tilde{f} \in W^{r_0,2}(B_R^+; \Lambda^k)$ and $\tilde{F} \in W^{r_1,2}(B_R^+; \Lambda^{k+1})$ satisfies,

$$\|\tilde{f}\|_{W^{r_0,2}(B_R^+; \Lambda^k)} \leq c_1 \left\{ \|f\|_{W^{r_0,2}(\Omega; \Lambda^k)} + \|F\|_{W^{r_1,2}(\Omega; \Lambda^k)} \right\}, \quad (6.26)$$

and

$$\|\tilde{F}\|_{W^{r_1,2}(B_R^+; \Lambda^k)} \leq c_2 \|F\|_{W^{r_1,2}(\Omega; \Lambda^k)}, \quad (6.27)$$

where $c_1 > 0$ is a constant, depending only on Φ and A .

Remark 6.24 Note that it follows from the statement of the lemma that it is possible to absorb the terms

$$\sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} p_{\alpha\beta} u^\alpha \psi^\beta \text{ and } \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} r_{\alpha\beta}^i \frac{\partial u^\alpha}{\partial x_i} \psi^\beta$$

in the terms $\sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha \psi_\alpha$ and the terms

$$\sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} q_{\alpha\beta}^i \frac{\partial \psi^\alpha}{\partial x_i} u^\beta$$

in the terms $\sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} b_\alpha^i \frac{\partial \psi^\alpha}{\partial x_i}$. We write this way just to make it easier to keep track of which terms are coming from where in the calculations in the proof.

Remark 6.25 The lemma essentially says that once we have flattened the boundary and froze the leading order coefficients at 0, we obtain a system which has the same form as (6.21), i.e of the form

$$\int_{B_R^+} \langle \bar{A}(du); d\psi \rangle + \int_{B_R^+} \langle \delta u; \delta \psi \rangle + \int_{B_R^+} \langle \tilde{f}; \psi \rangle - \int_{B_R^+} \langle \tilde{F}; d\psi \rangle,$$

where \bar{A} satisfies Legendre-Hadamard condition and $\tilde{f} \in L^2(B_R^+; \Lambda^k)$, $\tilde{F} \in L^2(B_R^+; \Lambda^{k+1})$, with L^2 norm of \tilde{f} and \tilde{F} being controlled by the L^2 norm of f and F and L^2 norm of F respectively, for every $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$, up to lower order terms and a second order term whose coefficient can be made arbitrarily small in C or $C^{0,\gamma}$, respectively. This is crucial for the boundary estimates since the boundary condition is well adapted to the operator $\delta(\bar{A}du) + d\delta u$, but not with the operator $-\text{div}(\tilde{A}\nabla u)$, which we used to derive the interior estimates.

Proof We start by noting that since $\Phi \in \text{Diff}^{r+2}(\overline{B_{R_0}}; \overline{W})$, respectively $\text{Diff}^{r+2,\gamma}(\overline{B_{R_0}}; \overline{W})$, we can assume that $D\Phi^{-1}(0) \in \text{SO}(n)$. By choosing $0 < R < R_0$ sufficiently small, we can always make the differences $D\Phi^{-1}(x) - D\Phi^{-1}(0)$ as small as we wish in C^{r+2-m} , respectively $C^{r+2-m,\gamma}$ norm for all $1 \leq m \leq r$. Also, since $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, respectively $C^{r+1,\gamma}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$, by choosing $0 < R < R_0$ small enough, we can make the difference $A(x_0) - A(x)$ as small as we wish in C^{r+1-m} , respectively $C^{r+1-m,\gamma}$ norm for all $0 \leq m \leq r$. Now choosing $\theta \in C_c^\infty(\Phi(B_R))$, since $\omega \in W_T^{1,2}(\Omega, \Lambda^k) \cap W^{r+1,2}(\Omega, \Lambda^k)$ is a weak solution of (6.21), we obtain, by lemma 6.22, that $\theta\omega$ satisfies (6.19).

Now for any $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$, extending ψ to a $W_{T,flat}^{1,2}(\mathbb{R}_+^n; \Lambda^k)$ map and taking the pullback by Φ^{-1} , we obtain $(\Phi^{-1})^* \psi \in W_T^{1,2}(\Omega, \Lambda^k)$. Hence, substituting in (6.19), we obtain that $u = \Phi^*(\theta\omega)$ satisfies,

$$\begin{aligned} \int_V \langle A(x) (d((\Phi^{-1})^* u); d((\Phi^{-1})^* \psi)) \rangle + \int_V \langle \delta((\Phi^{-1})^* u); \delta((\Phi^{-1})^* \psi) \rangle \\ + \int_V \langle \theta f - d\theta \lrcorner F; (\Phi^{-1})^* \psi \rangle - \int_V \langle \theta F; d((\Phi^{-1})^* \psi) \rangle \\ + \int_V \langle g; (\Phi^{-1})^* \psi \rangle - \int_V \langle d\theta \lrcorner \omega; \delta((\Phi^{-1})^* \psi) \rangle = 0, \end{aligned} \quad (6.28)$$

for every $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$, where $V = \Omega \cap (\Phi(B_R)) = \Phi(B_R^+)$ and g is given by (6.20).

Now we handle the terms one at a time. The last term, i.e

$$\int_V \langle d\theta \lrcorner \omega; \delta((\Phi^{-1})^* \psi) \rangle,$$

can be rewritten, after substituting the expression for $\delta((\Phi^{-1})^* \psi)$ and using the change of

variable formula as,

$$\int_V \langle d\theta \lrcorner \omega; \delta \left((\Phi^{-1})^* \psi \right) \rangle = \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} b_\alpha^i \frac{\partial \psi^\alpha}{\partial x_i} + \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha^1 \psi_\alpha,$$

where the functions b_α^i, a_α^1 depends on ω, Φ^{-1} and its first derivatives and first derivatives of θ . Indeed, these terms are components of ω multiplied with derivatives of θ and Φ^{-1} . Since θ is smooth, Φ^{-1} is C^{r+2} , respectively $C^{r+2,\gamma}$, $b_\alpha^i \in W^{r+1,2}$ for every $i = 1, \dots, n$ and every $\alpha \in \mathcal{T}^k$ with the estimates

$$\|b_\alpha^i\|_{W^{r+1,2}(B_R^+)} \leq c \|\omega\|_{W^{r+1,2}(\Omega, \Lambda^k)}, \quad \text{for all } i = 1, \dots, n \text{ and for all } \alpha \in \mathcal{T}^k,$$

for some constant $c > 0$, depending only on Φ and θ . But the choice of θ depends only on the choice of R , which is determined by Φ and A . So the constant depends on Φ and A .

Similarly, after substituting the expression for $(\Phi^{-1})^* \psi$ and the expression for g from (6.20) and using change of variables formula, we can write

$$\int_V \langle g; (\Phi^{-1})^* \psi \rangle = \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha^2 \psi_\alpha,$$

where the functions a_α^2 are components of ω and its first derivatives (coming from the expression for g), multiplied with components of A and their first derivatives and first derivatives of θ and Φ^{-1} . Taking $a_\alpha = a_\alpha^1 + a_\alpha^2$, this implies the estimate

$$\|a_\alpha\|_{W^{r,2}(B_R^+)} \leq c \|\omega\|_{W^{r+1,2}(\Omega, \Lambda^k)}, \quad \text{for all } \alpha \in \mathcal{T}^k,$$

for some constant $c > 0$, depending only on A, Φ and θ .

Once again, by similar argument as above, we can write,

$$\int_V \langle \theta f - d\theta \lrcorner F; (\Phi^{-1})^* \psi \rangle = \int_{B_R^+} \langle \tilde{f}; \psi \rangle,$$

where components of \tilde{f} are components of f and F , multiplied with first derivatives of θ and Φ^{-1} . Thus the estimate

$$\|\tilde{f}\|_{W^{r_0,2}(B_R^+; \Lambda^k)} \leq c_1 \left\{ \|f\|_{W^{r_0,2}(\Omega; \Lambda^k)} + \|F\|_{W^{r_1,2}(\Omega; \Lambda^{k+1})} \right\},$$

also holds with a constant $c_1 > 0$ which depends only on Φ and A .

Similarly, we can write,

$$\int_V \langle \theta F; d \left((\Phi^{-1})^* \psi \right) \rangle = \int_{B_R^+} \langle \tilde{F}; d\psi \rangle,$$

where components of \tilde{F} are components of F , multiplied with first derivatives of θ and Φ^{-1} . Thus the estimate

$$\|\tilde{F}\|_{W^{r_1,2}(B_R^+; \Lambda^k)} \leq c_2 \|F\|_{W^{r_1,2}(\Omega; \Lambda^{k+1})},$$

also holds.

Now it only remains to show that we can write

$$\begin{aligned}
& \int_V \langle A(x) \left(d((\Phi^{-1})^* u) ; d((\Phi^{-1})^* \psi) \right) \rangle + \int_V \langle \delta \left((\Phi^{-1})^* u \right) ; \delta \left((\Phi^{-1})^* \psi \right) \rangle \\
&= \int_{B_R^+} \langle \bar{A}(du) ; d\psi \rangle + \int_{B_R^+} \langle \delta u ; \delta \psi \rangle + \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} p_{\alpha\beta} u^\alpha \psi^\beta \\
&\quad + \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left(q_{\alpha\beta}^i \frac{\partial \psi^\alpha}{\partial x_i} u^\beta + r_{\alpha\beta}^i \frac{\partial u^\alpha}{\partial x_i} \psi^\beta \right) + \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} s_{\alpha\beta}^{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_j}.
\end{aligned}$$

Once again substituting the expressions for $(\Phi^{-1})^*$ and using change of variable formula, we note that for any constant coefficient matrix \bar{A} , we can always write it this form where the functions $p_{\alpha\beta}, q_{\alpha\beta}^i, r_{\alpha\beta}^i \in C^r(\overline{B_R^+})$, respectively $C^{r,\gamma}(\overline{B_R^+})$, since their components are multiplication of components of A , up to first order derivatives of θ and up to second order derivatives of Φ^{-1} , and $s_{\alpha\beta}^{ij} \in C^{r+1}(\overline{B_R^+})$, respectively $C^{r+1,\gamma}(\overline{B_R^+})$, since their components are multiplication of components of A , up to first order derivatives of θ and Φ^{-1} . But C^r , respectively $C^{r,\gamma}$ norm of $s_{\alpha\beta}^{ij}$ need not be small. So to prove the lemma, we just need to show that it is possible to choose a constant coefficient matrix \bar{A} , which satisfies Legendre-Hadamard condition such that we can make $\|s_{\alpha\beta}^{ij}\|_{C^r(\overline{B_R^+})}$, respectively $\|s_{\alpha\beta}^{ij}\|_{C^{r,\gamma}(\overline{B_R^+})}$, as small as we wish.

To show this, set

$$\tilde{\Phi}(y) = (D\Phi^{-1}(0)) y \quad \text{for every } y \in B_R.$$

Now note that the coefficient of the term with derivatives of both u and ψ , after using the change of variable formula for the difference

$$\int_V \langle A(x) \left(d((\Phi^{-1})^* u) ; d((\Phi^{-1})^* \psi) \right) \rangle - \int_{\tilde{\Phi}(B_R^+)} \langle A(x_0) \left(d((\tilde{\Phi})^* u) ; d((\tilde{\Phi})^* \psi) \right) \rangle,$$

can be made arbitrarily small in the C^r , respectively $C^{r,\gamma}$ norm, since they contain the differences $D\Phi^{-1}(x) - D\Phi^{-1}(0)$ and $A(x_0) - A(x)$. The same is true for the difference

$$\int_V \langle \delta \left((\Phi^{-1})^* u \right) ; \delta \left((\Phi^{-1})^* \psi \right) \rangle - \int_{\tilde{\Phi}(B_R^+)} \langle \delta \left((\tilde{\Phi})^* u \right) ; \delta \left((\tilde{\Phi})^* \psi \right) \rangle.$$

Since $\det(D\tilde{\Phi}) = \det(D\Phi^{-1}(0)) = 1$, by change of variable formula, we have,

$$\int_{\tilde{\Phi}(B_R^+)} \langle \delta \left((\tilde{\Phi})^* u \right) ; \delta \left((\tilde{\Phi})^* \psi \right) \rangle = \int_{B_R^+} \langle \delta u ; \delta \psi \rangle.$$

We denote $T = D\Phi^{-1}(0)$ and set

$$\bar{A} = (T^{-1})^* \circ A(x_0) \circ (T)^*.$$

Then, for any $\xi \in \Lambda^k$, we obtain,

$$T^* (\bar{A}\xi) = A(x_0) (T^*\xi). \quad (6.29)$$

Now, we have,

$$\int_{\tilde{\Phi}(B_R^+)} \langle A(x_0) \left(d((\tilde{\Phi})^* u) ; d((\tilde{\Phi})^* \psi) \right) \rangle = \int_{\tilde{\Phi}(B_R^+)} \langle A(x_0) \left(\tilde{\Phi}^*(du) \right) ; \tilde{\Phi}^*(d\psi) \rangle.$$

But $\tilde{\Phi}^*$ is the same as T^* . Hence, we obtain, using (6.29) and change of variable formula,

$$\int_{\tilde{\Phi}(B_R^+)} \langle A(x_0) \left(d((\tilde{\Phi})^* u) ; d((\tilde{\Phi})^* \psi) \right) \rangle = \int_{\tilde{\Phi}(B_R^+)} \langle (\tilde{\Phi}^* \bar{A}(du)) ; \tilde{\Phi}^*(d\psi) \rangle = \int_{B_R^+} \langle \bar{A}(du) ; d\psi \rangle.$$

Thus, it only remains to show that \bar{A} satisfies a Legendre-Hadamard condition. Now, for any $a \in \Lambda^1, b \in \Lambda^k$, we have

$$\begin{aligned} \langle \bar{A}(a \wedge b) ; a \wedge b \rangle &= \langle (T^{-1})^* \circ A(x_0) \circ (T)^*(a \wedge b) ; a \wedge b \rangle \\ &= \langle (T^{-1})^* \circ A(x_0) \circ (T)^*(a \wedge b) ; (T^{-1})^* \circ (T)^*(a \wedge b) \rangle \\ &= (T^{-1})^* (\langle A(x_0) \circ (T)^*(a \wedge b) ; (T)^*(a \wedge b) \rangle) \\ &= (T^{-1})^* (\langle A(x_0) (T^* a \wedge T^* b) ; (T^* a \wedge T^* b) \rangle). \end{aligned}$$

Since $(T^{-1})^*$, T^* are both bijective and $\langle A(x_0) (T^* a \wedge T^* b) ; (T^* a \wedge T^* b) \rangle \geq \gamma_0 |T^* a \wedge T^* b|^2$, there exists a $\tilde{\gamma}_0 > 0$ such that,

$$\langle \bar{A}(a \wedge b) ; a \wedge b \rangle \geq \tilde{\gamma}_0 |a \wedge b|^2 \quad \text{for all } a \in \Lambda^1, b \in \Lambda^k. \quad (6.30)$$

This completes the proof. ■

Theorem 6.26 ($W^{2,2}$ regularity up to the boundary) *Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $A \in C^1(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma_0 |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma_0 > 0$ for all $x \in \Omega$. Also let $f \in L^2(\Omega; \Lambda^k)$ and $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,

$$\int_{\Omega} \langle A(x) d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0, \quad (6.31)$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$. Then $\omega \in W^{2,2}(\Omega; \Lambda^k)$ and satisfies the estimate

$$\|\omega\|_{W^{2,2}(\Omega; \Lambda^k)} \leq c \left\{ \|\omega\|_{L^2(\Omega; \Lambda^k)} + \|f\|_{L^2(\Omega; \Lambda^k)} \right\},$$

where the constant $c > 0$ depends only on A, λ, γ_0 and Ω .

Proof We only need to prove the boundary estimate, since we have already shown the interior regularity results. Also, using a partition of unity for the boundary, it is enough to prove the result in a neighbourhood of a boundary point $x_0 \in \partial\Omega$. But using lemma 6.23 with $r_0 = r = 0$

and $F = 0$, it is enough to prove that $u \in W^{2,2}(B_R^+; \Lambda^k)$, where R is chosen as in lemma 6.23 and $u \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$ vanishes in a neighbourhood of the curved part of the boundary of B_R^+ and satisfies (6.22) for all $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$.

We use Nirenberg's difference quotients method. We recall the difference quotient operator

$$\tau_{h,s}u(x) = \frac{1}{h} \{u(x + he_s) - u(x)\}.$$

Fix $1 \leq s \leq n-1$. For $\psi \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$, we define

$$\tilde{\psi}(x) = \psi(x - he_s), \quad \text{for all } x \in B_R^+,$$

where e_s is the unit vector in the s -th coordinate direction and $h \in \mathbb{R}$. Then we have $\tilde{\psi} \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$. Plugging this as a test function in (6.22) and using the change of variables formula, we deduce,

$$\begin{aligned} \int_{B_R^+} \langle \bar{A}(du(x + he_s)); d\psi \rangle + \int_{B_R^+} \langle \delta u(x + he_s); \delta \psi \rangle + \int_{B_R^+} \langle \tilde{f}(x + he_s); \psi \rangle \\ + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0, \end{aligned} \quad (6.32)$$

where

$$\begin{aligned} I_1 &= \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha(x + he_s) \psi^\alpha, \\ I_2 &= \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} b_\alpha^i(x + he_s) \frac{\partial \psi^\alpha}{\partial x_i}, \\ I_3 &= \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} p_{\alpha\beta}(x + he_s) u^\alpha(x + he_s) \psi^\beta, \\ I_4 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} q_{\alpha\beta}^i(x + he_s) \frac{\partial \psi^\alpha}{\partial x_i} u^\beta(x + he_s), \\ I_5 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} r_{\alpha\beta}^i(x + he_s) \frac{\partial u^\alpha}{\partial x_i}(x + he_s) \psi^\beta, \\ I_6 &= \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} s_{\alpha\beta}^{ij}(x + he_s) \frac{\partial u^\alpha}{\partial x_i}(x + he_s) \frac{\partial \psi^\beta}{\partial x_j}. \end{aligned}$$

Subtracting the (6.22) from (6.32) and dividing by h , we obtain,

$$\begin{aligned} \int_{B_R^+} \langle \bar{A}(\tau_{h,s}du); d\psi \rangle + \int_{B_R^+} \langle \tau_{h,s}\delta u; \delta \psi \rangle + \int_{B_R^+} \langle \tau_{h,s}\tilde{f}; \psi \rangle \\ + I'_1 + I'_2 + I'_3 + I'_4 + I'_5 + I'_6 = 0, \end{aligned} \quad (6.33)$$

where

$$\begin{aligned}
I'_1 &= \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} a_\alpha) \psi^\alpha, \\
I'_2 &= \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} b_\alpha^i) \frac{\partial \psi^\alpha}{\partial x_i}, \\
I'_3 &= \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} p_{\alpha\beta}(x + h e_s) (\tau_{h,s} u^\alpha) \psi^\beta + \int_{B_R^+} (\tau_{h,s} p_{\alpha\beta}) u^\alpha \psi^\beta \right\}, \\
I'_4 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} q_{\alpha\beta}^i(x + h e_s) \frac{\partial \psi^\alpha}{\partial x_i} (\tau_{h,s} u^\beta) + \int_{B_R^+} (\tau_{h,s} q_{\alpha\beta}^i) \frac{\partial \psi^\alpha}{\partial x_i} u^\beta \right\}, \\
I'_5 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} r_{\alpha\beta}^i(x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \psi^\beta + \int_{B_R^+} (\tau_{h,s} r_{\alpha\beta}^i) \frac{\partial u^\alpha}{\partial x_i} \psi^\beta \right\}, \\
I'_6 &= \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} s_{\alpha\beta}^{ij}(x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \frac{\partial \psi^\beta}{\partial x_j} + \int_{B_R^+} (\tau_{h,s} s_{\alpha\beta}^{ij}) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_j} \right\}.
\end{aligned}$$

Since $1 \leq s \leq n-1$, $\tau_{h,s} u \in W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$ and hence can be used as a test function in (6.33). Plugging this and by Gaffney inequality and Gårding inequality (6.3) and noting that \bar{A} has constant coefficients, we deduce,

$$\begin{aligned}
\int_{B_R^+} |\nabla(\tau_{h,s} u)|^2 &\leq c_{01} \left\{ \int_{B_R^+} |(\tau_{h,s} u)|^2 + \int_{B_R^+} |d(\tau_{h,s} u)|^2 + \int_{B_R^+} |\delta(\tau_{h,s} u)|^2 \right\} \\
&\leq c_{01} \int_{B_R^+} |(\tau_{h,s} u)|^2 + c_{02} \int_{B_R^+} \langle \bar{A} d(\tau_{h,s} u); d(\tau_{h,s} u) \rangle + \int_{B_R^+} \langle \delta(\tau_{h,s} u); \delta(\tau_{h,s} u) \rangle \\
&\leq c_{01} \int_{B_R^+} |(\tau_{h,s} u)|^2 - c_{02} \left\{ \int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \right\} \\
&\leq c'_{01} \int_{B_R^+} |\nabla u|^2 - c_{02} \left\{ \int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \right\},
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} a_\alpha) (\tau_{h,s} u)^\alpha, \\
J_2 &= \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} b_\alpha^i) \frac{\partial (\tau_{h,s} u)^\alpha}{\partial x_i}, \\
J_3 &= \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} p_{\alpha\beta} (x + h e_s) (\tau_{h,s} u^\alpha) (\tau_{h,s} u)^\beta + \int_{B_R^+} (\tau_{h,s} p_{\alpha\beta}) u^\alpha (\tau_{h,s} u)^\beta \right\}, \\
J_4 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} q_{\alpha\beta}^i (x + h e_s) \frac{\partial (\tau_{h,s} u)^\alpha}{\partial x_i} (\tau_{h,s} u)^\beta + \int_{B_R^+} (\tau_{h,s} q_{\alpha\beta}^i) \frac{\partial (\tau_{h,s} u)^\alpha}{\partial x_i} u^\beta \right\}, \\
J_5 &= \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} r_{\alpha\beta}^i (x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) (\tau_{h,s} u)^\beta + \int_{B_R^+} (\tau_{h,s} r_{\alpha\beta}^i) \frac{\partial u^\alpha}{\partial x_i} (\tau_{h,s} u)^\beta \right\}, \\
J_6 &= \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} s_{\alpha\beta}^{ij} (x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \frac{\partial (\tau_{h,s} u)^\beta}{\partial x_j} + \int_{B_R^+} (\tau_{h,s} s_{\alpha\beta}^{ij}) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial (\tau_{h,s} u)^\beta}{\partial x_j} \right\}.
\end{aligned}$$

Now we want to estimate the terms $\int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle$ and J_1, J_2, J_3, J_4, J_5 and J_6 . We start with estimating $\int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle$.

$$\begin{aligned}
\left| \int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle \right| &\leq \left| \int_{B_R^+} \langle \tau_{h,s} \tilde{f}; \tau_{h,s} u \rangle \right| \leq \left| \int_{B_R^+} \langle \tilde{f}; \tau_{-h,s} (\tau_{h,s} u) \rangle \right| \leq \int_{B_R^+} |\langle \tilde{f}; \tau_{-h,s} (\tau_{h,s} u) \rangle| \\
&\leq \varepsilon \int_{B_R^+} |\tau_{-h,s} (\tau_{h,s} u)|^2 + c_{03} \int_{B_R^+} |\tilde{f}|^2 \leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{03} \int_{B_R^+} |\tilde{f}|^2.
\end{aligned}$$

Estimate of J_1

$$\begin{aligned}
\left| \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} a_\alpha) (\tau_{h,s} u)^\alpha \right| &\leq \left| \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha \tau_{-h,s} (\tau_{h,s} u^\alpha) \right| \leq \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} |a_\alpha \tau_{-h,s} (\tau_{h,s} u^\alpha)| \\
&\leq \varepsilon \int_{B_R^+} |\tau_{-h,s} (\tau_{h,s} u)|^2 + c_{04} \int_{B_R^+} \sum_{\alpha \in \mathcal{T}^k} |a_\alpha|^2 \\
&\leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{04} \|\omega\|_{W^{1,2}(\Omega, \Lambda^k)}^2.
\end{aligned}$$

Estimate of J_2

$$\begin{aligned}
\left| \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} (\tau_{h,s} b_\alpha^i) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} \right| &\leq \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} \left| (\tau_{h,s} b_\alpha^i) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} \right| \\
&\leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{05} \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} |(\tau_{h,s} b_\alpha^i)|^2 \\
&\leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{06} \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} |\nabla b_\alpha^i|^2 \\
&\leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{07} \|\omega\|_{W^{1,2}(\Omega, \Lambda^k)}^2.
\end{aligned}$$

Estimate of J_3

$$\begin{aligned}
&\left| \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} p_{\alpha\beta}(x + h e_s) (\tau_{h,s} u^\alpha) (\tau_{h,s} u)^\beta + \int_{B_R^+} (\tau_{h,s} p_{\alpha\beta}) u^\alpha (\tau_{h,s} u)^\beta \right\} \right| \\
&\leq \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \left| \int_{B_R^+} p_{\alpha\beta}(x + h e_s) (\tau_{h,s} u^\alpha) (\tau_{h,s} u)^\beta \right| + \left| \int_{B_R^+} (\tau_{h,s} p_{\alpha\beta}) u^\alpha (\tau_{h,s} u)^\beta \right| \right\}
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{\alpha, \beta \in \mathcal{T}^k} \left| \int_{B_R^+} p_{\alpha\beta}(x + h e_s) (\tau_{h,s} u^\alpha) (\tau_{h,s} u)^\beta \right| &\leq \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} |p_{\alpha\beta}(x + h e_s) (\tau_{h,s} u^\alpha) (\tau_{h,s} u)^\beta| \\
&\leq c_{08} M_p \int_{B_R^+} |(\tau_{h,s} u)|^2 + c_{08} \int_{B_R^+} |(\tau_{h,s} u)|^2 \\
&\leq c_{09} \int_{B_R^+} |\nabla u|^2,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\alpha, \beta \in \mathcal{T}^k} \left| \int_{B_R^+} (\tau_{h,s} p_{\alpha\beta}) u^\alpha (\tau_{h,s} u)^\beta \right| &\leq \sum_{\alpha, \beta \in \mathcal{T}^k} c_{10}^{\alpha\beta} \left\{ \int_{B_R^+} |(\tau_{h,s} p_{\alpha\beta}) u^\alpha|^2 + \int_{B_R^+} |\tau_{h,s} u^\beta|^2 \right\} \\
&\leq \sum_{\alpha, \beta \in \mathcal{T}^k} c_{10}^{\alpha\beta} \left\{ \int_{B_R^+} |p_{\alpha\beta}(\tau_{-h,s} u^\alpha)|^2 + \int_{B_R^+} |\tau_{h,s} u^\beta|^2 \right\} \\
&\leq c_{10} M_p \int_{B_R^+} |\nabla u|^2 + c_{11} \int_{B_R^+} |\nabla u|^2 \\
&\leq c_{12} \int_{B_R^+} |\nabla u|^2,
\end{aligned}$$

where $M_p = \max_{\alpha, \beta \in \mathcal{T}^k} \|p_{\alpha\beta}\|_{C(\overline{B_R^+})}$. Hence, combining the last two estimates, we obtain,

$$|J_3| \leq (c_{09} + c_{12}) \int_{B_R^+} |\nabla u|^2.$$

Estimate of J_4

$$\begin{aligned} & \left| \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} q_{\alpha\beta}^i(x + he_s) \frac{\partial (\tau_{h,s} u)^\alpha}{\partial x_i} (\tau_{h,s} u^\beta) + \int_{B_R^+} (\tau_{h,s} q_{\alpha\beta}^i) \frac{\partial (\tau_{h,s} u)^\alpha}{\partial x_i} u^\beta \right\} \right| \\ & \leq \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} \left| q_{\alpha\beta}^i(x + he_s) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} (\tau_{h,s} u^\beta) \right| + \int_{B_R^+} \left| (\tau_{h,s} q_{\alpha\beta}^i) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} u^\beta \right| \right\} \end{aligned}$$

Now,

$$\sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left| q_{\alpha\beta}^i(x + he_s) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} (\tau_{h,s} u^\beta) \right| \leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{13} M_q \int_{B_R^+} |\nabla u|^2,$$

and

$$\begin{aligned} \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left| (\tau_{h,s} q_{\alpha\beta}^i) \frac{\partial (\tau_{h,s} u^\alpha)}{\partial x_i} u^\beta \right| & \leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{14,i}^{\alpha\beta} \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left| (\tau_{h,s} q_{\alpha\beta}^i) u^\beta \right|^2 \\ & \leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{15} M_q \int_{B_R^+} |\nabla u|^2, \end{aligned}$$

where $M_q = \max_{\substack{i=1, \dots, n, \\ \alpha, \beta \in \mathcal{T}^k}} \|q_{\alpha\beta}^i\|_{C(\overline{B_R^+})}$, since

$$\int_{B_R^+} \left| (\tau_{h,s} q_{\alpha\beta}^i) u^\beta \right|^2 = \int_{B_R^+} \left| q_{\alpha\beta}^i (\tau_{-h,s} u^\beta) \right|^2 \quad \text{for all } i = 1, \dots, n, \text{ for all } \alpha, \beta \in \mathcal{T}^k.$$

Combining the last two estimates, we obtain,

$$|J_4| \leq 2\varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + (c_{13} + c_{15}) M_q \int_{B_R^+} |\nabla u|^2.$$

Estimate of J_5

$$\begin{aligned} & \left| \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} r_{\alpha\beta}^i(x + he_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) (\tau_{h,s} u)^\beta + \int_{B_R^+} (\tau_{h,s} r_{\alpha\beta}^i) \frac{\partial u^\alpha}{\partial x_i} (\tau_{h,s} u)^\beta \right\} \right| \\ & \leq \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} \left| r_{\alpha\beta}^i(x + he_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) (\tau_{h,s} u)^\beta \right| + \int_{B_R^+} \left| (\tau_{h,s} r_{\alpha\beta}^i) \frac{\partial u^\alpha}{\partial x_i} (\tau_{h,s} u)^\beta \right| \right\}. \end{aligned}$$

Now,

$$\sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left| r_{\alpha\beta}^i(x + he_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) (\tau_{h,s} u)^\beta \right| \leq \varepsilon \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{16} M_r \int_{B_R^+} |\nabla u|^2,$$

and

$$\begin{aligned}
\sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left| (\tau_{h,s} r_{\alpha\beta}^i) \frac{\partial u^\alpha}{\partial x_i} (\tau_{h,s} u^\beta) \right| &\leq \varepsilon \sum_{i=1, \dots, n} \int_{B_R^+} \left| (\tau_{h,s} r_{\alpha\beta}^i) (\tau_{h,s} u^\beta) \right|^2 + c_{17} \int_{B_R^+} |\nabla u|^2 \\
&\leq \varepsilon \sum_{i=1, \dots, n} \int_{B_R^+} |r_{\alpha\beta}^i|^2 \left| \tau_{-h,s} (\tau_{h,s} u^\beta) \right|^2 + c_{17} \int_{B_R^+} |\nabla u|^2 \\
&\leq \varepsilon c_{18} M_r \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{17} \int_{B_R^+} |\nabla u|^2,
\end{aligned}$$

where $M_r = \max_{i=1, \dots, n, \alpha, \beta \in \mathcal{T}^k} \|r_{\alpha\beta}^i\|_{C(\overline{B_R^+})}$. Combining the last two estimates, we deduce,

$$|J_5| \leq \varepsilon (1 + c_{18} M_r) \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + (c_{16} M_r + c_{17}) \int_{B_R^+} |\nabla u|^2.$$

Estimate of J_6 This one is trickier. As before, we deduce,

$$\begin{aligned}
&\left| \sum_{i,j=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} s_{\alpha\beta}^{ij}(x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} + \int_{B_R^+} (\tau_{h,s} s_{\alpha\beta}^{ij}) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right\} \right| \\
&\leq \sum_{i,j=1, \dots, n} \sum_{\alpha, \beta \in \mathcal{T}^k} \left\{ \int_{B_R^+} \left| s_{\alpha\beta}^{ij}(x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right| + \int_{B_R^+} \left| (\tau_{h,s} s_{\alpha\beta}^{ij}) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right| \right\}.
\end{aligned}$$

Now for any $i, j = 1, \dots, n$, any $\alpha, \beta \in \mathcal{T}^k$, we have,

$$\begin{aligned}
&\int_{B_R^+} \left| s_{\alpha\beta}^{ij}(x + h e_s) \left(\tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right) \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right| \\
&\leq \|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)} \left(\int_{B_R^+} \left| \tau_{h,s} \frac{\partial u^\alpha}{\partial x_i} \right|^2 \right)^{\frac{1}{2}} \left(\int_{B_R^+} \left| \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2.
\end{aligned}$$

We also have,

$$\begin{aligned}
\int_{B_R^+} \left| \left(\tau_{h,s} s_{\alpha\beta}^{ij} \right) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right| &\leq \varepsilon \int_{B_R^+} \left| \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right|^2 + c_{19} \int_{B_R^+} \left| \left(\tau_{h,s} s_{\alpha\beta}^{ij} \right) \frac{\partial u^\alpha}{\partial x_i} \right|^2 \\
&\leq \varepsilon c_{20} \int_{B_R^+} \left| \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right|^2 + c_{19} \int_{B_R^+} \left| s_{\alpha\beta}^{ij} \tau_{-h,s} \left(\frac{\partial u^\alpha}{\partial x_i} \right) \right|^2 \\
&\leq \varepsilon c_{20} \int_{B_R^+} \left| \frac{\partial (\tau_{h,s} u^\beta)}{\partial x_j} \right|^2 + c_{19} \|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)} \int_{B_R^+} \left| \tau_{-h,s} \left(\frac{\partial u^\alpha}{\partial x_i} \right) \right|^2 \\
&\leq \varepsilon c_{20} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{19} \|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)} \int_{B_R^+} \left| \tau_{h,s} \left(\frac{\partial u^\alpha}{\partial x_i} \right) \right|^2 \\
&\leq \varepsilon c_{20} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{19} \|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2,
\end{aligned}$$

for any $i, j = 1, \dots, n$, any $\alpha, \beta \in \mathcal{T}^k$. Since $\|s_{\alpha\beta}^{ij}\|_{L^\infty(B_R^+)}$ can be made arbitrarily small, these two estimates imply,

$$|J_6| \leq \varepsilon c_{21} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2.$$

Plugging in all these estimates, we deduce,

$$\int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 \leq \varepsilon c_{22} \int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 + c_{23} \int_{B_R^+} |\tilde{f}|^2 + c_{24} \int_{B_R^+} |\nabla u|^2 + c_{25} \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2.$$

Choosing ε small enough such that $1 - \varepsilon c_{22} > 0$, we obtain, after transposing,

$$\int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 \leq c_{26} \int_{B_R^+} |\tilde{f}|^2 + c_{27} \int_{B_R^+} |\nabla u|^2 + c_{28} \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2.$$

Since

$$\int_{B_R^+} |\nabla u|^2 \leq c_{29} \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2$$

and

$$\int_{B_R^+} |\tilde{f}|^2 \leq c_{30} \|f\|_{L^2(\Omega; \Lambda^k)}^2,$$

we obtain,

$$\int_{B_R^+} |\nabla (\tau_{h,s} u)|^2 \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\} \quad \text{for all } s = 1, \dots, n-1. \quad (6.34)$$

This implies, for all $s = 1, \dots, n-1, l = 1, \dots, n$, and for all $I \in \mathcal{T}^k$,

$$\left\| \frac{\partial}{\partial x_s} \left(\frac{\partial u^I}{\partial x_l} \right) \right\|_{L^2(B_R^+)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}.$$

Since weak derivatives commute, this implies that for any $I \in \mathcal{T}^k$ and for all $p, q = 1, \dots, n$,

$(p, q) \neq (n, n)$, there exists a constant c such that,

$$\left\| \frac{\partial^2(u^I)}{\partial x_p \partial x_q} \right\|_{L^2(B_R^+)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}. \quad (6.35)$$

Now to prove $u \in W^{2,2}(B_R^+; \Lambda^k)$, it only remains to show that there is a constant c such that for all $I \in \mathcal{T}^k$,

$$\left\| \frac{\partial^2(u^I)}{\partial x_n \partial x_n} \right\|_{L^2(B_R^+)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}. \quad (6.36)$$

To show this, we define the linear map $\tilde{A} : \Lambda^{k+1} \rightarrow \Lambda^k$ by,

$$\tilde{A} = (\pi^{ext, k+1})^T \circ \bar{A} \circ \pi^{ext, k+1} + (\pi^{int, k-1})^T \circ \pi^{int, k-1},$$

where $\pi^{ext, k+1}, \pi^{int, k-1}$ are the projection maps defined in chapter 3 and $(\cdot)^T$ denotes the transpose. By lemma 6.15, \tilde{A} satisfies,

$$\langle \tilde{A}(a \otimes b); a \otimes b \rangle \geq \gamma_1 |a|^2 |b|^2 \quad \text{for every } a \in \mathbb{R}^n, b \in \mathbb{R}^{\binom{n}{k}},$$

for some constant $\gamma_1 > 0$. We also define the maps $\tilde{A}^{pq} : \mathbb{R}^{\binom{n}{k}} \rightarrow \mathbb{R}^{\binom{n}{k}}$ for every $p, q = 1, \dots, n$, by the identities,

$$\sum_{\alpha, \beta \in \mathcal{T}^k} \tilde{A}_{\alpha\beta}^{pq} \xi^\alpha \xi^\beta = \langle \tilde{A}(e_p \otimes \xi); e_q \otimes \xi \rangle \quad \text{for every } \xi \in \mathbb{R}^{\binom{n}{k}}.$$

Also, note that this in particular implies,

$$\langle \tilde{A}^{nn} \xi; \xi \rangle = \sum_{\alpha, \beta \in \mathcal{T}^k} \tilde{A}_{\alpha\beta}^{nn} \xi^\alpha \xi^\beta = \langle \tilde{A}(e_n \otimes \xi); e_n \otimes \xi \rangle \geq \gamma_1 |\xi|^2,$$

for every $\xi \in \mathbb{R}^{\binom{n}{k}}$. Thus, \tilde{A}^{nn} is invertible.

Now we start with equation (6.22) and rewriting, we obtain,

$$\begin{aligned} & \int_{B_R^+} \sum_{\alpha, \beta \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \frac{\partial \psi^\beta}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \frac{\partial \psi^\beta}{\partial x_n} \right) \\ &= - \int_{B_R^+} \sum_{\substack{p, q=1, \dots, n \\ (p, q) \neq (n, n) \\ \alpha, \beta \in \mathcal{T}^k}} \tilde{A}_{\alpha\beta}^{pq} \frac{\partial u^\alpha}{\partial x_p} \frac{\partial \psi^\beta}{\partial x_q} - \int_{B_R^+} \langle \tilde{f}; \psi \rangle - \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} a_\alpha \psi_\alpha - \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \int_{B_R^+} b_\alpha^i \frac{\partial \psi^\alpha}{\partial x_i} \\ & \quad - \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} p_{\alpha\beta} u^\alpha \psi^\beta - \sum_{i=1}^n \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} \left(q_{\alpha\beta}^i \frac{\partial \psi^\alpha}{\partial x_i} u^\beta + r_{\alpha\beta}^i \frac{\partial u^\alpha}{\partial x_i} \psi^\beta \right) \\ & \quad - \sum_{\substack{i, j=1, \dots, n \\ (i, j) \neq (n, n)}} \sum_{\alpha, \beta \in \mathcal{T}^k} \int_{B_R^+} s_{\alpha\beta}^{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_j} = 0, \end{aligned}$$

This implies,

$$\begin{aligned}
& \sum_{\beta \in \mathcal{T}^k} \int_{B_R^+} \left[\sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \right) \right] \frac{\partial \psi^\beta}{\partial x_n} \\
&= \sum_{\beta \in \mathcal{T}^k} \int_{B_R^+} \left[\sum_{\substack{p,q=1,\dots,n \\ (p,q) \neq (n,n)}} \sum_{\alpha \in \mathcal{T}^k} \frac{\partial}{\partial x_q} \left(\tilde{A}_{\alpha\beta}^{pq} \frac{\partial u^\alpha}{\partial x_p} \right) - \tilde{f}^\beta - a_\beta + \sum_{i=1}^n \frac{\partial b_\beta^i}{\partial x_i} - \sum_{\alpha \in \mathcal{T}^k} p_{\alpha\beta} u^\alpha \right. \\
&\quad \left. + \sum_{i=1}^n \sum_{\alpha \in \mathcal{T}^k} \left(\frac{\partial}{\partial x_i} (q_{\beta\alpha}^i u^\alpha) - r_{\alpha\beta}^i \frac{\partial u^\alpha}{\partial x_i} \right) + \sum_{\substack{i,j=1,\dots,n \\ (i,j) \neq (n,n)}} \sum_{\alpha \in \mathcal{T}^k} \frac{\partial}{\partial x_j} \left(s_{\alpha\beta}^{ij} \frac{\partial u^\alpha}{\partial x_i} \right) \right] \psi^\beta = 0.
\end{aligned}$$

By definition of the weak derivatives, this means,

$$\frac{\partial}{\partial x_n} \left(\sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \right) \right) \in L^2(B_R^+) \quad \text{for every } \beta \in \mathcal{T}^k,$$

with the estimate

$$\left\| \frac{\partial}{\partial x_n} \left(\sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \right) \right) \right\|_{L^2(B_R^+)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}, \quad (6.37)$$

for every $\beta \in \mathcal{T}^k$. For every $h > 0$, let us denote by $B_R^{+,h}$ the set

$$B_R^{+,h} = \{x \in B_R^+ : \text{dist}(x, \Gamma) > h\}.$$

Fix any $\beta \in \mathcal{T}^k$. Then (6.37) implies, for every $h > 0$,

$$\left\| \tau_{h,n} \left(\sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \right) \right) \right\|_{L^2(B_R^{+,h})} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}.$$

By discrete Leibniz rule, we have,

$$\tau_{h,n} \left(\sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial u^\alpha}{\partial x_n} \right) \right) = \sum_{\alpha \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial \tau_{h,n} u^\alpha}{\partial x_n} + \tau_{h,n} (s_{\alpha\beta}^{nn}) \frac{\partial u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial \tau_{h,n} u^\alpha}{\partial x_n} \right).$$

for every $h > 0$. Since $s_{\alpha\beta}^{nn} \in C^1(\overline{B_R^+})$, $\tau_{h,n}(s_{\alpha\beta}^{nn}) \frac{\partial u^\alpha}{\partial x_n} \in L^2(B_R^+)$. Thus, we obtain, after summing over all $\beta \in \mathcal{T}^k$,

$$\left\| \sum_{\alpha, \beta \in \mathcal{T}^k} \left(\tilde{A}_{\alpha\beta}^{nn} \frac{\partial \tau_{h,n} u^\alpha}{\partial x_n} + s_{\alpha\beta}^{nn} \frac{\partial \tau_{h,n} u^\alpha}{\partial x_n} \right) \right\|_{L^2(B_R^{+,h})} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\},$$

for every $h > 0$. Now since \tilde{A}^{nn} is invertible and $\|s_{\alpha\beta}^{nn}\|_{L^\infty}$ can be made arbitrarily small, we get,

$$c_1 \int_{B_R^{+,h}} \left| \tau_{h,n} \left(\frac{\partial u}{\partial x_n} \right) \right|^2 - \varepsilon \int_{B_R^{+,h}} \left| \tau_{h,n} \left(\frac{\partial u}{\partial x_n} \right) \right|^2 \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\},$$

for every $h > 0$. Thus,

$$\limsup_{h \rightarrow 0} \int_{B_R^{+,h}} \left| \tau_{h,n} \left(\frac{\partial u}{\partial x_n} \right) \right|^2 \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}.$$

This implies

$$\left\| \frac{\partial^2 u}{\partial x_n \partial x_n} \right\|_{L^2(B_R^+; \Lambda^k)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}.$$

This shows $u \in W^{2,2}(B_R^+; \Lambda^k)$ with the estimate

$$\|D^2 u\|_{L^2(B_R^+; \Lambda^k)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}. \quad (6.38)$$

Since we also have the easy estimate that

$$\|u\|_{W^{1,2}(B_R^+; \Lambda^k)} \leq c \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)},$$

combining we obtain,

$$\|u\|_{W^{2,2}(B_R^+; \Lambda^k)} \leq c \left\{ \|\omega\|_{W^{1,2}(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}. \quad (6.39)$$

Since $u \in W^{2,2}(B_R^+; \Lambda^k)$, by Gagliardo-Nirenberg inequality and Young's inequality, we obtain,

$$\begin{aligned} \|u\|_{W^{1,2}(B_R^+; \Lambda^k)} &= \|u\|_{L^2(B_R^+; \Lambda^k)} + \|\nabla u\|_{L^2(B_R^+; \Lambda^k)} \\ &\leq c_1 \|u\|_{L^2(B_R^+; \Lambda^k)} + c \|D^2 u\|_{L^2(B_R^+; \Lambda^k)}^{\frac{1}{2}} \|u\|_{L^2(B_R^+; \Lambda^k)}^{\frac{1}{2}} \\ &\leq c\varepsilon \|D^2 u\|_{L^2(B_R^+; \Lambda^k)} + (c\varepsilon + c_1) \|u\|_{L^2(B_R^+; \Lambda^k)}. \end{aligned}$$

Choosing ε small enough to absorb the norm of $D^2 u$ on the left side of (6.39) and estimating L^2 norm of u by L^2 norm of ω , we obtain the desired estimate for u . This finishes the proof. ■

Remark 6.27 *The trick of using Gagliardo-Nirenberg inequality and Young's inequality can be applied to (6.38) as well to obtain the estimate,*

$$\|D^2 u\|_{L^2(B_R^+; \Lambda^k)} \leq c \left\{ \|\omega\|_{L^2(\Omega; \Lambda^k)}^2 + \|f\|_{L^2(\Omega; \Lambda^k)}^2 \right\}.$$

Higher regularity

For any integer $r \geq 0$, $\phi \in W^{r+2,2}(B_R^+; \Lambda^k) \cap W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$ implies $\frac{\partial \phi}{\partial x_s} \in W^{r+1,2}(B_R^+; \Lambda^k) \cap W_{T,flat}^{1,2}(B_R^+; \Lambda^k)$ for every $s = 1, \dots, n-1$ (but not for $s = n$). Indeed, since $\nu \wedge \phi = e_n \wedge \phi = 0$ on Γ , we have, $\phi_I = 0$ on Γ for all $I \in \mathcal{T}^k, n \notin I$. This implies $\frac{\partial \phi_I}{\partial x_s} = 0$ on Γ , for every $s = 1, \dots, n-1$. But this means $\nu \wedge \frac{\partial \phi}{\partial x_s} = e_n \wedge \frac{\partial \phi}{\partial x_s} = 0$ on Γ for every $s = 1, \dots, n-1$. Using

this fact and using lemma 6.23 we can iterate the same procedure to prove the higher regularity results, which we state below and omit the proof.

Theorem 6.28 ($W^{r+2,2}$ regularity up to the boundary) *Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $r \geq 0$ be an integer and $A \in C^{r+1}(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma > 0$ for all $x \in \Omega$. Also let $f \in W^{r,2}(\Omega; \Lambda^k)$ and let $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0,$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$. Then $\omega \in W^{r+2,2}(\Omega; \Lambda^k)$ and satisfies the estimate

$$\|\omega\|_{W^{r+2,2}(\Omega; \Lambda^k)} \leq c \left\{ \|\omega\|_{L^2(\Omega; \Lambda^k)} + \|f\|_{W^{r,2}(\Omega; \Lambda^k)} \right\},$$

where the constant $c > 0$ depends only on A, λ, γ_0 and Ω .

Before commenting on up to the boundary regularity in the scale of $W^{r,p}$ and $C^{r,\alpha}$ spaces, we first want to show a consequence of Theorem 6.26.

Theorem 6.29 *Let $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set. Let $A \in C^1(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k$$

for some constant $\gamma > 0$ for all $x \in \Omega$. Also let $f \in L^2(\Omega; \Lambda^k)$ and let $\lambda \in \mathbb{R}$. Let $\omega \in W_T^{1,2}(\Omega, \Lambda^k)$ be a weak solution of the following,

$$\int_{\Omega} \langle A(x)d\omega, d\phi \rangle + \int_{\Omega} \langle \delta\omega, \delta\phi \rangle + \lambda \int_{\Omega} \langle \omega, \phi \rangle + \int_{\Omega} \langle f, \phi \rangle = 0, \quad (6.40)$$

for all $\phi \in W_T^{1,2}(\Omega; \Lambda^k)$. Then $\omega \in W^{2,2}(\Omega; \Lambda^k)$ is also a solution to the following boundary value problem for the Hodge-type system:

$$\begin{cases} \delta(A(x)d\omega) + d\delta\omega = \lambda\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \\ \nu \wedge \delta\omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{H})$$

Proof The fact that $\omega \in W^{2,2}(\Omega; \Lambda^k)$ is immediately implied by theorem 6.26. Integrating by parts, we obtain,

$$\int_{\Omega} \langle \delta(A(x)d\omega) + d\delta\omega; \phi \rangle - \int_{\partial\Omega} (\langle d\omega; \nu \wedge \phi \rangle + \langle \nu \wedge \delta\omega; \phi \rangle) = \int_{\Omega} \langle \lambda\omega + f; \phi \rangle,$$

for all $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. Thus taking $\phi \in C_c^\infty(\Omega, \Lambda^k)$ we have,

$$\delta(A(x)d\omega) + d\delta\omega = \lambda\omega + f + \delta F \quad \text{in } \Omega.$$

But this implies that the integral on the boundary vanish separately. But since $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$, $\nu \wedge \phi = 0$. Hence we obtain,

$$\int_{\partial\Omega} \langle \nu \wedge \delta\omega; \phi \rangle = 0$$

for any $\phi \in W_T^{1,2}(\Omega, \Lambda^k)$. We now show that this identity is valid for any $u \in W^{1,2}(\Omega, \Lambda^k)$ as well. Extending ν as a C^1 function inside Ω and using the identity

$$u = \nu \wedge (\nu \lrcorner u) + \nu \lrcorner (\nu \wedge u),$$

we deduce, for any $u \in W^{1,2}(\Omega, \Lambda^k)$,

$$\int_{\partial\Omega} \langle \nu \wedge \delta\omega; u \rangle = \int_{\partial\Omega} \langle \nu \wedge \delta\omega; \nu \wedge (\nu \lrcorner u) \rangle + \int_{\partial\Omega} \langle \nu \wedge \delta\omega; \nu \lrcorner (\nu \wedge u) \rangle = \int_{\partial\Omega} \langle \nu \wedge \delta\omega; \nu \wedge (\nu \lrcorner u) \rangle = 0,$$

since $\nu \wedge (\nu \lrcorner u) \in W_T^{1,2}(\Omega, \Lambda^k)$. Since $u \in W^{1,2}(\Omega, \Lambda^k)$ is arbitrary, this implies $\nu \wedge \delta\omega = 0$ on $\partial\Omega$ and finishes the proof. ■

Now up to the boundary regularity in the scale of $W^{r,p}(p \neq 2)$ and $C^{r,\alpha}$ spaces can be obtained straight away once it can be shown that the boundary conditions satisfy the so-called ‘Agmon-Douglis-Nirenberg’s complementing condition’ (cf. [2]), also called the ‘L-condition’ or ‘Lopatinski-Shapiro condition’, with respect to the system of partial differential operators, which in our case is strongly elliptic. However, the verification of these conditions seems extremely tedious in this generality.

However, there are two important special cases where the regularity up to the boundary results in $W^{r,p}$ and $C^{r,\alpha}$ spaces are long-known. One of them is when $k = 1$ and $n = 3$. In this case, by virtue of the vector calculus identity

$$\operatorname{div} \circ \operatorname{curl} \equiv 0,$$

the regularity result follows from the regularity result for the scalar elliptic equation. This trick does not generalize to $n \neq 3$ or $k \neq 1$. Although a recent argument by Dacorogna-Gangbo-Kneuss [27] seems to work in any dimension as long as $k = 1$, once again by reducing the problem to a single scalar elliptic equation. The other one is the case when A is the identity matrix. In this case, the regularity result for this system follows from then regularity theory of the Hodge Laplacian, which is classical. Below we briefly sketch the arguments for proving regularity in this case.

Comments on regularity results for the Hodge Laplacian The regularity theory of the Hodge Laplacian with relative or absolute boundary condition is well-known and classical (see chapter 7 in Morrey [53]). The crucial point is, when $A \equiv \mathbf{I}$, the system essentially decouples

into a number of scalar Laplace operators. We use admissible coordinate systems to flatten the boundary. Although the transformed system in a boundary neighbourhood of a point x_0 in the boundary of the half-space need not have constant coefficients and is of the same general form as (\mathcal{H}) , the essential difference is that in this case it can be ensured that $A(x_0) = \mathbf{I}$. But $\delta d + d\delta$, i.e the Hodge Laplacian is precisely the componentwise scalar Laplacian. Also, the boundary condition $e_n \wedge \omega = 0$ implies that $\omega_I = 0$ on flat part of the boundary for every $I \in \mathcal{T}^k$ such that $n \notin I$. But this implies $\frac{\partial \omega_I}{\partial x_s} = 0$ for every $s = 1, \dots, n-1$ and for every $I \in \mathcal{T}^k$ such that $n \notin I$. This together with the boundary condition $e_n \wedge \delta \omega = 0$ implies that $\frac{\partial \omega_I}{\partial x_n} = 0$ for every $I \in \mathcal{T}^k$ such that $n \in I$. So the whole system decouples and gets reduced to $\binom{n}{k}$ scalar Poisson problem with lower order terms, out of which $\binom{n-1}{k}$ number of equations, corresponding the components ω_I where $n \notin I$, has zero Dirichlet boundary conditions and the other $\binom{n-1}{k-1}$ number of equations, corresponding the components ω_I where $n \in I$, has zero Neumann boundary conditions. Also note that the lower order terms need not necessarily decouple, but that does not affect the regularity results. Regularity theory thus follows from the results about scalar Poisson equations. In chapter 7 of [53], Morrey proves the regularity results by using explicitly writing a representation formula for each component of the solution using the Green and Neumann function for the Laplacian.

So the methods in both these cases, i.e the case of time-harmonic Maxwell's equation and the Hodge Laplacian case, ultimately relies on the reduction of the system to one or more scalar elliptic equations and thus are inapplicable to deduce the regularity for our case, which is truly a system and not reducible to the scalar case. Also, the Agmon-Douglis-Nirenberg complementing conditions are hard to verify. However, it seems possible to obtain the regularity estimates directly by deriving a Cacciopoli type inequality and estimates in Campanato spaces, which we shall not discuss in this thesis. (see [60]).

6.5 Main theorems

Now we are in a position to prove the central theorems of this chapter.

Theorem 6.30 (Maxwell type system with tangential data) *Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Also let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy either the Legendre-Hadamard condition or the Legendre condition. Then there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$, such that if $\lambda \notin \sigma$, then for any $f \in W^{r,2}(\Omega, \Lambda^k)$ satisfying $\delta f = 0$, there exists a unique solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_T^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,*

$$\begin{cases} \delta(A(x)d\omega) = \lambda\omega + f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{P})$$

Also for each $\sigma_i \in \sigma$ there exist non-trivial weak solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ to the following

boundary value problem,

$$\begin{cases} \delta(A(x)d\alpha) = \sigma_i \alpha & \text{in } \Omega, \\ \delta\alpha = 0 & \text{in } \Omega, \\ \nu \wedge \alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{EV})$$

and the space of solutions to (\mathcal{EV}) is finite-dimensional for any $\sigma_i \in \sigma$.

Proof Since $f \in W^{r,2}(\Omega, \Lambda^k)$ satisfies $\delta f = 0$, we can find $F \in W^{r+1,2}(\Omega, \Lambda^{k+1})$ such that

$$\begin{cases} dF = 0 & \text{and } \delta F = f & \text{in } \Omega, \\ \nu \wedge F = 0 & & \text{on } \partial\Omega. \end{cases}$$

Thus we replace f by δF and then existence of weak solution part is exactly Theorem 6.11. Now since, by once again replacing δF by f , any weak solution to (\mathcal{P}) satisfies (6.10), applying Theorem 6.28, we obtain the $W^{r+2,2}$ regularity. Also, in the same way, Theorem 6.28 implies that any solution to (\mathcal{EV}) is in $W^{m,2}$ for any integer $m \geq 0$, which by Sobolev embedding implies the C^∞ regularity and establishes the theorem. ■

Remark 6.31 Note that if A has constant coefficients and satisfies Legendre-Hadamard condition or if A satisfies the Legendre condition, then ρ can be taken as zero. In other words, in these two cases, for every $\lambda \geq 0$, \mathcal{EV} has only trivial solution and \mathcal{P} can always be solved for any $f \in W^{r,2}(\Omega, \Lambda^k)$ satisfying $\delta f = 0$.

Now we present an important consequence of the theorem above.

Theorem 6.32 (Maxwell type operator with full Dirichlet data) Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Also let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k+1}, \Lambda^{k+1}))$ be such that any one of the following two conditions (H1) (H2) holds.

(H1) A satisfy the Legendre-Hadamard condition and there is no non-trivial solutions $\alpha \in W_T^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\alpha) = 0 & \text{in } \Omega, \\ \delta\alpha = 0 & \text{in } \Omega, \\ \nu \wedge \alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{EV}_0)$$

(H2) A satisfy the Legendre condition.

Then for any $\omega_0 \in W^{r+2,2}(\Omega, \Lambda^k)$ and any $f \in W^{r,2}(\Omega, \Lambda^k)$ such that $\delta f = 0$ in the sense of distributions, there exists a solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\omega) = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_D)$$

Remark 6.33 Once again, if $A(x)$ is a constant matrix satisfying the Legendre-Hadamard ellipticity condition then we can have,

$$\int_{\Omega} \langle Adv, dv \rangle \geq \gamma \int_{\Omega} \langle dv, dv \rangle,$$

forcing every solution to (EV0) to be trivial. Hence in that case we can always solve (\mathcal{P}_D) .

Proof With the Legendre-Hadamard condition, if (EV0) does not admit a non-trivial solution, then this implies the problem,

$$\begin{cases} \delta(A(x)d\bar{\omega}) = f - \delta(A(x)d\omega_0) \text{ in } \Omega, \\ \delta\bar{\omega} = 0 \text{ in } \Omega, \\ \nu \wedge \bar{\omega} = 0 \text{ on } \partial\Omega, \end{cases} \quad (\mathcal{P}_V)$$

has an unique solution $\bar{\omega} \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_T^{1,2}(\Omega, \Lambda^k)$ by theorem 6.30. Now since $\nu \wedge (-\bar{\omega}) = 0$ on $\partial\Omega$, we can find $v \in W^{r+3,2}(\Omega, \Lambda^{k-1})$ (cf. lemma 8.11 in [21]) such that $dv = -\bar{\omega}$ on $\partial\Omega$. Then setting $\omega = \omega_0 + \bar{\omega} + dv$, we have,

$$\delta(A(x)d\omega) = \delta(A(x)(d\omega_0 + d\bar{\omega} + ddv)) = \delta(A(x)d\omega_0) + \delta(A(x)d\bar{\omega}) = f \quad \text{in } \Omega.$$

Also, since $dv = -\bar{\omega}$ on $\partial\Omega$, we have $\omega = \omega_0$ on $\partial\Omega$. Hence $\omega \in W^{r+2,2}(\Omega, \Lambda^k)$ is a solution to (\mathcal{P}_D) . This proves the result.

With the Legendre ellipticity assumption, the only modification to the above proof is to note is that because of the stronger ellipticity assumption, we have,

$$\int_{\Omega} \langle Adv, dv \rangle \geq \gamma \int_{\Omega} |dv|^2.$$

Hence the (EV0) can not admit a non-trivial solution. This establishes the theorem. ■

The last two theorems immediately yield the corresponding dual versions.

Theorem 6.34 (Maxwell type system with normal data) Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Also let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $A \in C^{r+1}(\bar{\Omega}; L(\Lambda^{k-1}, \Lambda^{k-1}))$ satisfy either the Legendre condition or there exists a constant γ_0 such that for every $x \in \Omega$, A satisfies,

$$\langle A(x)(a \lrcorner b); a \lrcorner b \rangle \geq \gamma_0 |a \lrcorner b|^2 \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k.$$

Then there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$, such that if $\lambda \notin \sigma$, then for any $f \in W^{r,2}(\Omega, \Lambda^k)$ satisfying $df = 0$, there exists a unique solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_N^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\delta\omega) = \lambda\omega + f \text{ in } \Omega, \\ d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{P}_N)$$

Also for each $\sigma_i \in \sigma$ there exists non-trivial weak solutions $\alpha \in C^\infty(\overline{\Omega}, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\delta\alpha) = \sigma_i\alpha \text{ in } \Omega, \\ d\alpha = 0 \text{ in } \Omega, \\ \nu \lrcorner \alpha = 0 \text{ on } \partial\Omega, \end{cases} \quad (\mathcal{EV}_N)$$

and the space of solutions to (\mathcal{EV}) is finite-dimensional for any $\sigma_i \in \sigma$.

Proof The proof is just a matter of Hodge duality. Define

$$\tilde{A} := (-1)^{(k-1)(n-k+1)} * \circ A \circ *,$$

where $*$ is the Hodge star operator. Now, we have, for any $a \in \Lambda^1, b \in \Lambda^{n-k}$,

$$\begin{aligned} \langle A(x)(a \lrcorner (*b)); a \lrcorner (*b) \rangle &= \langle A(x)(* (a \wedge (**b))); *(a \wedge (**b)) \rangle \\ &= \langle A(x)(* (a \wedge b)); *(a \wedge b) \rangle \\ &= (-1)^{(k-1)(n-k+1)} \langle *(A(x)(* (a \wedge b))); a \wedge b \rangle \\ &= \langle \tilde{A}(x)(a \wedge b); a \wedge b \rangle \end{aligned}$$

Hence, we have, for any $a \in \Lambda^1, b \in \Lambda^{n-k}$,

$$\begin{aligned} \langle \tilde{A}(x)(a \wedge b); a \wedge b \rangle &\geq \gamma_0 |a \lrcorner (*b)|^2 \\ &= \gamma_0 \left| (-1)^{n(k-1)} * (a \wedge (**b)) \right|^2 \\ &= \gamma_0 \left| (-1)^{n(k-1)+(k)(n-k)} * (a \wedge b) \right|^2 \\ &\geq \gamma |a \wedge b|^2, \end{aligned}$$

for some positive constant $\gamma > 0$, by invertibility of the Hodge star operator. But this proves that the linear map $\tilde{A} : \Lambda^{n-k+1} \rightarrow \Lambda^{n-k+1}$ satisfies the Legendre-Hadamard condition. Also, it is clear that $\tilde{A} \in C^{r+1}(\overline{\Omega}; L(\Lambda^{n-k+1}, \Lambda^{n-k+1}))$. Also, the hypotheses on f clearly imply $*f \in W^{r,2}(\Omega, \Lambda^{n-k})$. Also, we have,

$$\delta(*f) = (-1)^{n(n-k-1)} * (d(* * f)) = (-1)^{n(n-k-1)+k(n-k)} * (df) = 0.$$

Now we claim that $\omega \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_N^{1,2}(\Omega, \Lambda^k)$ is a solution to (\mathcal{P}_N) if and only if $*\omega \in W^{r+2,2}(\Omega, \Lambda^{n-k}) \cap W_T^{1,2}(\Omega, \Lambda^{n-k})$ satisfies

$$\begin{cases} \delta(\tilde{A}(x)d(*\omega)) = \lambda(*\omega) + *f \text{ in } \Omega, \\ \delta(*\omega) = 0 \text{ in } \Omega, \\ \nu \wedge (*\omega) = 0 \text{ on } \partial\Omega. \end{cases}$$

Indeed, taking Hodge star on both sides, we obtain,

$$*(\delta(\tilde{A}(x)d(*\omega))) = \lambda(*\omega) + **f$$

which implies

$$\begin{aligned} (-1)^{(k-1)(n-k+1)} * (\delta((-1)^{n(k-1)} * (A(x)(\delta\omega))) &= (-1)^{k(n-k)}(\lambda\omega + f) \\ \Rightarrow (-1)^{(k-1)(n-k+1)} (-1)^{n(k-1)+n(n-k)} **d(**(A(x)(\delta\omega))) &= (-1)^{k(n-k)}(\lambda\omega + f) \\ \Rightarrow (-1)^{(k-1)(n-k+1)} (-1)^{n(n-1)} (-1)^{k(n-k)} (-1)^{(k-1)(n-k+1)} d(A(x)(\delta\omega)) &= (-1)^{k(n-k)}(\lambda\omega + f) \\ \Rightarrow (-1)^{k(n-k)} d(A(x)(\delta\omega)) &= (-1)^{k(n-k)}(\lambda\omega + f) \\ \Rightarrow d(A(x)(\delta\omega)) &= \lambda\omega + f. \end{aligned}$$

Also,

$$0 = *\delta(*\omega) = (-1)^{n(n-k-1)} **d(*\omega) = (-1)^{n(n-k-1)+k(n-k)+(k+1)(n-k-1)} d\omega,$$

and

$$0 = *(\nu \wedge (*\omega)) = (-1)^{n(k-1)}(\nu \lrcorner \omega).$$

The previous calculation also shows that the same goes true for the eigenvalue problem (\mathcal{EV}_N) . Hence, theorem 6.30 implies the result and finishes the proof. ■

The same Hodge duality argument proves

Theorem 6.35 (Dual Maxwell operator with full Dirichlet data) *Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Also let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $A \in C^{r+1}(\overline{\Omega}; L(\Lambda^{k-1}, \Lambda^{k-1}))$ be such that any one of the following two conditions (H1) (H2) holds.*

(H1) *For every $x \in \Omega$, A satisfies,*

$$\langle A(x)(a \lrcorner b); a \lrcorner b \rangle \geq \gamma_0 |a \lrcorner b|^2 \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k.$$

and there is no non-trivial solutions $\alpha \in W_N^{1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\delta\alpha) = 0 & \text{in } \Omega, \\ d\alpha = 0 & \text{in } \Omega, \\ \nu \lrcorner \alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{EVD})$$

(H2) *A satisfy the Legendre condition.*

Then for any $\omega_0 \in W^{r+2,2}(\Omega, \Lambda^k)$ and any $f \in W^{r,2}(\Omega, \Lambda^k)$ such that $df = 0$ in the sense of distributions, there exists a solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\delta\omega) = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_{D,dual})$$

6.6 Applications of Linear theory

As a consequence of the existence and regularity theory, we also deduce an existence theorem (cf. theorem 6.36) for the following first order linear boundary value problem,

$$\begin{cases} d(A(x)(\omega)) = f & \text{and} & \delta(B(x)(\omega)) = g & \text{in } \Omega, \\ \nu \wedge A(x)\omega = \nu \wedge \omega_0 & & & \text{on } \partial\Omega. \end{cases} \quad (6.41)$$

This existence result for (6.41) is also new and generalizes the existing results on the well-studied special case (cf. [21]),

$$\begin{cases} d\omega = f & \text{and} & \delta\omega = g & \text{in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 & & & \text{on } \partial\Omega. \end{cases}$$

6.6.1 Div-Curl type first order linear system

Theorem 6.36 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Given two maps $A, B \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$ such that A is invertible, $A^{-1} \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$ and BA^{-1} be such that any one of the following two conditions (H1) (H2) holds.*

(H1) $BA^{-1} : \Omega \rightarrow L(\Lambda^k, \Lambda^k)$ satisfy the Legendre-Hadamard condition and there is no non-trivial weak solutions $\alpha \in W_T^{1,2}(\Omega, \Lambda^{k-1})$ to the following boundary value problem,

$$\begin{cases} \delta(BA^{-1}(x)d\alpha) = 0 & \text{in } \Omega, \\ \delta\alpha = 0 & \text{in } \Omega, \\ \nu \wedge \alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{EV0})$$

(H2) $BA^{-1} : \Omega \rightarrow L(\Lambda^k, \Lambda^k)$ satisfy the Legendre condition.

Then for any $\omega_0 \in W^{r+2,2}(\Omega, \Lambda^k)$, for any two forms $f \in W^{r,2}(\Omega, \Lambda^{k+1})$ and $g \in W^{r,2}(\Omega, \Lambda^{k-1})$ such that, $df = 0$, $\delta g = 0$ in Ω and $\nu \wedge d\omega_0 = \nu \wedge f$ on $\partial\Omega$, there exists an unique solution $\omega \in W^{r+1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\omega) = f & \text{and} & \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \wedge A(x)\omega = \nu \wedge \omega_0 & & & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_1)$$

Proof We prove only the case $2 \leq k \leq n-1$. The case $k=1$ is much easier. The hypotheses on f imply (cf. theorem 8.16 in [21]) that there exists $F \in W^{r+1,2}(\Omega, \Lambda^k)$ such that

$$\begin{aligned} dF &= f & \text{in } \Omega, \\ F &= \omega_0 & \text{on } \partial\Omega. \end{aligned}$$

Now, now since $BA^{-1} \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$, we can use theorem 6.26 to find a solution $\alpha \in W^{r+2,2}(\Omega, \Lambda^{k-1})$ such that

$$\begin{aligned}\delta(BA^{-1}d\alpha) &= g - \delta(BA^{-1}F) && \text{in } \Omega, \\ \delta\alpha &= 0 && \text{in } \Omega, \\ \nu \wedge \alpha &= 0 && \text{on } \partial\Omega.\end{aligned}$$

Now we define,

$$\omega = A^{-1}(d\alpha + F).$$

Note that, since $A^{-1} \in C^{r+1}(\Omega; L(\Lambda^k, \Lambda^k))$, $\omega \in W^{r+1,2}(\Omega; L(\Lambda^k, \Lambda^k))$. Then,

$$A\omega = d\alpha + F,$$

and

$$B\omega = BA^{-1}A\omega = BA^{-1}(d\alpha + F)$$

Hence, we have,

$$\begin{aligned}d(A(x)\omega) &= d(d\alpha + F) = dF = f && \text{in } \Omega, \\ \delta(B(x)\omega) &= \delta(BA^{-1}(x)(d\alpha + F)) = g && \text{in } \Omega, \\ \nu \wedge A\omega &= \nu \wedge (d\alpha + F) = \nu \wedge \omega_0 && \text{on } \partial\Omega,\end{aligned}$$

as $\nu \wedge d\alpha = 0$ (since $\nu \wedge \alpha = 0$) and $F = \omega_0$ on $\partial\Omega$. ■

Again we also have the dual version.

Theorem 6.37 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Given two maps $A, B \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$ such that B is invertible, $B^{-1} \in C^{r+1}(\overline{\Omega}; L(\Lambda^k, \Lambda^k))$ and AB^{-1} be such that any one of the following two conditions (H1) (H2) holds.*

(H1) $AB^{-1} : \Omega \rightarrow L(\Lambda^k, \Lambda^k)$ satisfies, for every $x \in \Omega$,

$$\langle AB^{-1}(x)(a \lrcorner b); a \lrcorner b \rangle \geq \gamma_0 |a \lrcorner b|^2 \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k,$$

for some $\gamma_0 > 0$ and there is no non-trivial weak solution $\alpha \in W_N^{1,2}(\Omega, \Lambda^{k+1})$ to the following boundary value problem,

$$\begin{cases} d(AB^{-1}(x)\delta\alpha) = 0 & \text{in } \Omega, \\ \delta\alpha = 0 & \text{in } \Omega, \\ \nu \lrcorner \alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{EV1})$$

(H2) $AB^{-1} : \Omega \rightarrow L(\Lambda^k, \Lambda^k)$ satisfy the Legendre condition.

Then for any $\omega_0 \in W^{r+2,2}(\Omega, \Lambda^k)$, for any two forms $f \in W^{r,2}(\Omega, \Lambda^{k+1})$ and $g \in W^{r,2}(\Omega, \Lambda^{k-1})$ such that, $df = 0$, $\delta g = 0$ in Ω and $\nu \lrcorner g = \nu \lrcorner \delta\omega_0$ on $\partial\Omega$, there exists an unique solution

$\omega \in W^{r+1,2}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x)\omega) = f & \text{and} & \delta(B(x)\omega) = g & \text{in } \Omega, \\ \nu \lrcorner B(x)\omega = \nu \lrcorner \omega_0 & & & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_2)$$

6.6.2 Hodge Laplacian type elliptic system

The regularity theory also enables us to solve a second order elliptic system.

Theorem 6.38 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $1 \leq k \leq n-1$ and $r \geq 0$ be integers. Let $A \in C^{r+1}(\Omega; L(\Lambda^{k+1}, \Lambda^{k+1}))$ satisfy,*

$$\langle A(x)(a \wedge b) ; a \wedge b \rangle \geq \gamma |a \wedge b|^2, \quad \text{for every } a \in \Lambda^1, b \in \Lambda^k, \text{ for all } x \in \Omega,$$

for some constant $\gamma > 0$. Then there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$, with no limit points except possibly $-\infty$, such that if $\lambda \notin \sigma$, then for any $\omega_0 \in W^{r+2,2}(\Omega, \Lambda^k)$ and any $f \in W^{r,2}(\Omega, \Lambda^k)$, there exists a solution $\omega \in W^{r+2,2}(\Omega, \Lambda^k)$ to the following boundary value problem:

$$\begin{cases} \delta(A(x)d\omega) + d\delta\omega = \lambda\omega + f & \text{in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 & \text{on } \partial\Omega, \\ \nu \wedge \delta\omega = \nu \wedge \delta\omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{H})$$

Also for each $\sigma_i \in \sigma$ there exists non-trivial weak solutions $\alpha \in C^\infty(\bar{\Omega}, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\alpha) + d\delta\alpha = \sigma_i\alpha & \text{in } \Omega, \\ \nu \wedge \alpha = 0 & \text{on } \partial\Omega, \\ \nu \wedge \delta\alpha = 0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{EV}_H)$$

and the space of solutions to (\mathcal{EV}_H) is finite-dimensional for any $\sigma_i \in \sigma$.

Proof We divide the proof in two steps.

Step 1 (Existence): The proof of existence of weak solutions is very similar to the arguments in Section 6.2, so we just sketch the arguments. We start by showing existence of weak solution for sufficiently large positive values of λ .

For a given $\lambda \in \mathbb{R}$, we define the bilinear operators $a_1 : W_T^{1,2}(\Omega; \Lambda^k) \times W_T^{1,2}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$, $a_2 : W_T^{1,2}(\Omega; \Lambda^k) \times W_T^{1,2}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ and $b_\lambda : W_T^{1,2}(\Omega; \Lambda^k) \times W_T^{1,2}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ by,

$$\begin{aligned} a_1(u, v) &= \int_{\Omega} \langle A(x)du, dv \rangle, \\ a_2(u, v) &= \int_{\Omega} \langle \delta u, \delta v \rangle, \\ b_\lambda(u, v) &= a_1(u, v) + a_2(u, v) + \lambda \int_{\Omega} \langle u, v \rangle. \end{aligned}$$

Clearly, $a_1(u, v), a_2(u, v)$ is continuous and so is $b_\lambda(u, v)$ for any $\lambda \in \mathbb{R}$, so we need only check the coercivity. Since $W_T^{1,2}(\Omega; \Lambda^k) \subset W_T^{d,2}(\Omega, \Lambda^k)$, by theorem 6.3, there exists constants $\lambda_0 > 0$ and λ_1 such that,

$$a_1(v, v) \geq \lambda_0 \|dv\|_{L^2}^2 - \lambda_1 \|v\|_{L^2}^2.$$

Then for any $\lambda \geq \lambda_1$, we have, by Gaffney inequality,

$$\begin{aligned} b_\lambda(v, v) &= a_1(v, v) + a_2(v, v) + \lambda \int_\Omega \langle v, v \rangle \\ &= a_1(v, v) + a_2(v, v) + \lambda \|v\|_{L^2}^2 \\ &\geq \lambda_0 \|dv\|_{L^2}^2 - \lambda_1 \|v\|_{L^2}^2 + \|\delta v\|_{L^2}^2 + \lambda \|v\|_{L^2}^2 \\ &= \lambda_0 \|dv\|_{L^2}^2 + \|\delta v\|_{L^2}^2 + (\lambda - \lambda_1) \|v\|_{L^2}^2 \\ &\geq \tilde{\lambda}_0 \left(\|dv\|_{L^2}^2 + \|\delta v\|_{L^2}^2 \right) \\ &\geq \tilde{\lambda}_0 \|v\|_{W^{1,2}}^2, \end{aligned}$$

where $\tilde{\lambda}_0 = \min\{\lambda_0, 1\} > 0$. Now Lax-Milgram theorem implies the existence of $\bar{w} \in W_T^{1,2}(\Omega; \Lambda^k)$ satisfying

$$b_\lambda(\bar{w}, \theta) = - \int_\Omega \langle g, \theta \rangle \quad \text{for all } \theta \in W_T^{1,2}(\Omega; \Lambda^k),$$

for any $g \in L^2(\Omega, \Lambda^k)$.

Now as in section 6.2, we can define a ‘solution operator’ $T_\lambda^- : (W_T^{1,2}(\Omega; \Lambda^k))^* \rightarrow W_T^{1,2}(\Omega; \Lambda^k)$ which is a bounded linear operator. Since $W_T^{1,2}(\Omega; \Lambda^k)$ embeds compactly in $L^2(\Omega; \Lambda^k)$, an analogue of lemma 6.8 holds and arguing as in theorem 6.10, we prove that there exists a constant $\rho \in \mathbb{R}$ and an at most countable set $\sigma \subset (-\infty, \rho)$ such that if $\lambda \notin \sigma$, the integro-differential equation,

$$\int_\Omega \langle A(x) d\bar{w}, d\theta \rangle + \langle \delta \bar{w}, \delta \theta \rangle + \lambda \int_\Omega \langle \bar{w}, \theta \rangle + \int_\Omega \langle g, \theta \rangle = 0,$$

for all $\theta \in W_T^{1,2}(\Omega; \Lambda^k)$, has a unique solution $\bar{w} \in W_T^{1,2}(\Omega; \Lambda^k)$. Moreover, the set σ does not have a limit point except possibly $-\infty$. If σ is infinite, then it is a non-increasing sequence $\{\lambda_i\}$ such that $\lambda_i \rightarrow -\infty$ as $i \rightarrow \infty$. Also, for every $\sigma_i \in \sigma$, there exists non-trivial solutions $\alpha \in W_T^{1,2}(\Omega; \Lambda^k)$, $\alpha \neq 0$ which solves the following integro-differential equation,

$$\int_\Omega \langle A(x) d\alpha, d\theta \rangle + \langle \delta \alpha, \delta \theta \rangle + \sigma_i \int_\Omega \langle \alpha, \theta \rangle = 0$$

for all $\theta \in W_T^{1,2}(\Omega; \Lambda^k)$. Moreover, the subspace of such solutions is finite dimensional.

Step 2 (Regularity): Now theorem 6.28 gives us the desired regularity, i.e it shows that $\alpha \in C^\infty(\bar{\Omega}, \Lambda^k)$ and $\bar{w} \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_T^{1,2}(\Omega; \Lambda^k)$ if $g \in W^{r,2}(\Omega, \Lambda^k)$. Integrating by parts, we immediately obtain that α is a solution to $(\mathcal{E}\mathcal{V}_H)$. Also, arguing as in theorem 6.29 we obtain that $\bar{w} \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_T^{1,2}(\Omega; \Lambda^k)$ satisfies,

$$\delta(A(x) d\bar{w}) + d\delta \bar{w} = \lambda \bar{w} + g \text{ in } \Omega,$$

and

$$\nu \wedge \bar{\omega} = 0 \quad \text{and} \quad \nu \wedge \delta \bar{\omega} = 0 \quad \text{in } \partial\Omega.$$

Taking $g = f + \lambda\omega_0 - \delta(A(x)d\omega_0) - d\delta\omega_0 \in W^{r,2}(\Omega, \Lambda^k)$ and setting $\omega = \bar{\omega} + \omega_0$, we immediately see that $\omega \in W^{r+2,2}(\Omega, \Lambda^k) \cap W_T^{1,2}(\Omega; \Lambda^k)$ is a solution to (\mathcal{H}) . This finishes the proof. ■

Chapter 7

Maxwell operator for k -forms: Nonlinear Case

7.1 Introduction

Semilinear theory The prototype of the semilinear problems for the Maxwell type operator concerns a power type nonlinearity. However, as the principal linear part of the operator controls only the exterior derivative, but not the full gradient of the solution, the natural space to derive existence results are various partial Sobolev spaces rather than the usual ones. Since these partial spaces do not embed into L^p spaces, in general the problem is considerably harder than the semilinear problems for scalar elliptic equations. For much of the same reason, sign of the nonlinearity plays a very crucial role. In section 7.2, the main example of the problems we shall treat is the following boundary value problem,

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + |\omega|^{p-2}\omega + f \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega, \end{cases} \quad (7.1)$$

with $2 \leq p < \infty$. The crucial point here is that in this case, the operator is monotone and coercive (if the problem has a variational structure, the energy functional is convex and coercive) as long as $\lambda \in \mathbb{R}$ is at a positive distance away from the spectrum of the linear operator in (6.2). In theorem 7.1, we shall show how standard monotone operator theory yields an existence theorem for (7.1) and slightly more general problems. However, it is important to note that the problem completely changes its character if $\lambda \in \mathbb{R}$ is not at a positive distance away from the spectrum of the linear operator in (6.2).

In section 7.3, we investigate the case when the sign of the nonlinearity is such that the energy functional is neither coercive nor convex. The prototype of the problem we are interested in is the following boundary value problem,

$$\begin{cases} \delta(A(x)(d\omega)) + |\omega|^{p-2}\omega = \lambda\omega \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (7.2)$$

Note also that it is crucial for our analysis that there is no source term on the right hand side ($f \equiv 0$) and the boundary value is also identically 0.

For the scalar case, i.e $k = 0$, the analogue to this problem is the well-known

$$\begin{cases} \Delta u + |u|^{p-2}u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

This one is extremely well-studied and for $2 < p < 2^*$, i.e the so-called subcritical semilinear problem, existence can be obtained for all values of λ . In general, these problems can not be approached by minimization techniques as it is a priori clear that the energy functional attains neither a global minimum nor a global maximum. Even for the scalar case, the relevant techniques are provided by critical point theory. In other words, to derive existence for these problems, we look for a non-trivial critical point of the energy functional. However, for the problem (7.2), every non-trivial critical point is a degenerate critical point and must have an infinite Morse index, due to the huge, infinite dimensional kernel of the linear operator $\delta(A(x)(d(\cdot)))$. This is an additional difficulty which is not present in the scalar case.

Due to these difficulties, we can resolve the problem only in the case where $\lambda \leq 0$, i. e in the real half line in the direction of the spectrum of the linear operator in (6.2). We develop the abstract critical point theory needed to analyze the problem, which uses the method of generalized Nehari manifold or ‘Nehari-Pankov’ manifold, essentially due to Pankov (see Szulkin-Weth [66] for a nice presentation). However, some modification of the method presented there is needed to handle our case, due to the additional obstacle that $W^{d,2,p}(\Omega; \Lambda^k)$ (cf. Definition 2.19 for definition of these spaces) does not embed compactly into $L^p(\Omega; \Lambda^k)$. These modifications were essentially worked out in Bartsch-Mederski [13], where they resolve the following prototype problem:

$$\begin{cases} \operatorname{curl} \operatorname{curl} \vec{u} + \lambda \vec{u} = |\vec{u}|^{p-2} \vec{u} \text{ in } \Omega, \\ \vec{\nu} \times \vec{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

in 3 dimensions. Note that since $\delta du = -\operatorname{curl} \operatorname{curl} u$, so the nonlinearity has the sign of the noncoercive case. We resolve the general case ((7.2) with slightly more general hypothesis on the nonlinearity) in theorem 7.5. The result in the generality we state here is new. Though the hypotheses on the nonlinearity and as such, the basic techniques do not differ much from the ones in [13], modifications are necessary to treat the case of the operator with Legendre-Hadamard type of ellipticity assumption.

Quasilinear theory The prototype problem for the quasilinear version of the Maxwell type operators for k -forms is the system

$$\begin{cases} \delta(A(x, d\omega)) = f \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega. \end{cases}$$

We prove existence of a weak solution to this system. When the system has a variational structure, existence can be deduced simply by using minimization techniques. In particular, theorem 3.64 can be applied. Here instead we prove this result by showing first the existence

of solutions to the related system

$$\begin{cases} \delta(A(x, d\omega)) = f \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases} \quad (7.3)$$

Both these results seem to be new. However, a related problem have received some attention in the past. The solutions of the system

$$\delta(\varrho(|\omega|^2)\omega) = 0 \quad \text{and} \quad d\omega = 0 \text{ in } \Omega,$$

$$\begin{cases} \delta(\varrho(|\omega|^2)\omega) = 0 & \text{and} & d\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \delta(\varrho(|\omega|^2)\omega) = 0 & \text{and} & d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner (\varrho(|\omega|^2)\omega) = 0 \text{ on } \partial\Omega, \end{cases}$$

are called ϱ -harmonic k -forms, ϱ -harmonic Dirichlet k -forms and ϱ -harmonic Neumann k -forms respectively. In a well-known paper ([70]) Uhlenbeck obtained interior $C^{1,\alpha}$ regularity results for ϱ -harmonic k -forms. Later in another widely known paper([35]), Hamburger showed the existence and up to the boundary $C^{1,\alpha}$ regularity for ϱ -harmonic Dirichlet and Neumann k -forms. To compare these result with the one presented here, it is useful to consider exact forms $\omega = d\alpha$ so that the condition $d\omega = 0$ is automatically satisfied and we can rewrite the system for a ϱ -harmonic Dirichlet k -form ω as the following system for α ,

$$\begin{cases} \delta(\varrho(|d\alpha|^2)d\alpha) = 0 \text{ in } \Omega, \\ \nu \wedge d\alpha = 0 \text{ on } \partial\Omega. \end{cases}$$

Now since $\nu \wedge \alpha = 0$ on $\partial\Omega$ implies $\nu \wedge d\alpha = 0$ on $\partial\Omega$, it is clear that for any solution α to the system (7.3) in the special case when $A(x, d\alpha) = \varrho(|d\alpha|^2)d\alpha$ and $f = 0$, $\omega = d\alpha$ is a ϱ -harmonic Dirichlet k -form. However, there is no such obvious connections of our results to the ϱ -harmonic Neumann k -forms.

7.2 Semilinear theory: Coercive case

There are two distinct classes of semilinear problems that are of interest. In this section, we shall deal with the coercive case. This is the easier case of semilinear equations, where the bilinear form associated with the problem, i.e the ‘energy functional’ is not indefinite.

7.2.1 Existence of weak solutions

Theorem 7.1 *Let $1 \leq k \leq n-1$, $2 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfy either the Legendre-Hadamard condition and is uniformly continuous*

or the Legendre condition and is bounded and measurable. Also let $B \in L^\infty(\Omega; L(\Lambda^k, \Lambda^{k+1}))$, $C \in L^\infty(\Omega; L(\Lambda^{k+1}, \Lambda^k))$ and $D \in L^\infty(\Omega; L(\Lambda^k, \Lambda^k))$ and let $\omega_0 \in W^{d,2,p}(\Omega, \Lambda^k)$, $F \in L^2(\Omega, \Lambda^{k+1})$ and $f \in L^{p'}(\Omega, \Lambda^k)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $\rho : \Omega \times \Lambda^k \rightarrow \Lambda^k$ be a map such that,

(N1) There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^k$,

$$|\rho(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^k$,

$$\langle \rho(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) For every $u, v \in W^{d,2,p}(\Omega, \Lambda^k)$,

$$\langle \rho(x, u(x)) - \rho(x, v(x)), u(x) - v(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Then there exists a constant $\tilde{\lambda}$ such that for any constant $\lambda \geq \tilde{\lambda}$, there exists a solution $\omega \in \omega_0 + W_T^{d,2,p}(\Omega, \Lambda^k)$ to the following integro-differential equation,

$$\begin{aligned} \int_{\Omega} [\langle A(x)d\omega, d\theta \rangle + \langle B(x)\omega, d\theta \rangle - \langle C(x)d\omega, \theta \rangle - \langle D(x)\omega, \theta \rangle] \\ + \lambda \int_{\Omega} \langle \omega, \theta \rangle + \int_{\Omega} \langle \rho(x, \omega), \theta \rangle + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0, \end{aligned}$$

for all $\theta \in W_T^{d,2,p}(\Omega, \Lambda^k)$. In other words, there exists a weak solution $\omega \in W^{d,2,p}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\omega) + \delta(B(x)\omega) + C(x)d\omega + D(x)\omega = \lambda\omega + \rho(x, \omega) + f + \delta F \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{P}_0)$$

Remark 7.2 (1) In particular, the theorem is true for $\rho(x, \omega) = |\omega|^{p-2}\omega$.

(2) The hypotheses (N1), (N2) and (N3) on the nonlinearity are satisfied if there exists a function $W : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ such that $\rho(x, \xi) = \nabla_{\xi} W(x, \xi)$ for a.e $x \in \Omega$ and $\xi \mapsto W(x, \xi)$ is convex for all $\xi \in \Lambda^k$ for a.e $x \in \Omega$ and there are constants $0 < c_1 < c_2$ such that $c_1 (|\xi|^p - 1) \leq W(x, \xi) \leq c_2 (|\xi|^p + 1)$ for all $\xi \in \Lambda^k$ for a.e $x \in \Omega$. In particular, $W(x, \xi) = \frac{1}{p}|\xi|^p$ satisfies the requirements.

(3) As the proof will show, if $B, C, D = 0$ and A either satisfies Legendre condition or is a constant matrix satisfying Legendre-Hadamard condition, then the constant $\tilde{\lambda}$ can be taken to be 0. Combined with the previous remark, this means, in particular, that the following boundary value problem,

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + |\omega|^{p-2}\omega + f + \delta F \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega, \end{cases}$$

admits a weak solution $\omega \in W^{d,2,p}(\Omega, \Lambda^k)$ for all $\lambda \geq 0$ and all boundary values $\omega_0 \in W^{d,2,p}(\Omega, \Lambda^k)$.

- (4) If $V \in L^\infty(\Omega)$ is positive and bounded away from zero, i.e there exists a constant $\alpha > 0$ such that $V(x) \geq \alpha > 0$ for a.e $x \in \Omega$, then $\rho(x, \xi) = V(x)\xi$ satisfies all the hypothesis of the theorem with $p = 2$. This implies, if A either satisfies Legendre condition or is a constant matrix satisfying Legendre-Hadamard condition and $V \in L^\infty(\Omega)$ be positive and bounded away from zero, then the linear boundary value problem,

$$\begin{cases} \delta(A(x)(d\omega)) = \lambda\omega + V(x)\omega + f + \delta F \text{ in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 \text{ on } \partial\Omega, \end{cases}$$

admits a weak solution $\omega \in W^{d,2}(\Omega, \Lambda^k)$ for all $\lambda \geq 0$ and all boundary values $\omega_0 \in W^{d,2}(\Omega, \Lambda^k)$. It is important to note that, as we have already remarked, though this problem is linear, it is not possible to handle this problem by the methods presented in the section for linear theory for sign-changing V or for negative values of λ since the term linear in ω is not a compact perturbation to the Maxwell operator.

Proof For a given $\lambda \in \mathbb{R}$ and a given $\omega_0 \in W^{d,2,p}(\Omega, \Lambda^k)$, we start by defining the bilinear operator $a : W^{d,2,p}(\Omega, \Lambda^k) \times W^{d,2,p}(\Omega, \Lambda^k) \rightarrow \mathbb{R}$ and the operator $a_{\lambda,p} : W_T^{d,2,p}(\Omega, \Lambda^k) \times W_T^{d,2,p}(\Omega, \Lambda^k) \rightarrow \mathbb{R}$ by,

$$\begin{aligned} a(u, v) &= \int_{\Omega} [\langle A(x)du, dv \rangle + \langle B(x)u, dv \rangle - \langle C(x)du, v \rangle - \langle D(x)u, v \rangle], \\ a_{\lambda,p}(u, v) &= a(u, v) + \lambda \int_{\Omega} \langle u, v \rangle + \int_{\Omega} \langle \rho(x, u + \omega_0), v \rangle. \end{aligned}$$

Clearly, $a_{\lambda,p} : W_T^{d,2,p}(\Omega, \Lambda^k) \times W_T^{d,2,p}(\Omega, \Lambda^k) \rightarrow \mathbb{R}$ is linear in the second variable but nonlinear in the first. Our plan is to use Minty-Browder theory of monotone operators (cf. theorem 3 in [18]). First note that both the operators are separately continuous in both variables in view of the following estimates,

$$\begin{aligned} |a_{\lambda,p}(u, v)| &\leq \|A\|_{L^\infty} \|du\|_{L^2} \|dv\|_{L^2} + \|B\|_{L^\infty} \|u\|_{L^2} \|dv\|_{L^2} + \|C\|_{L^\infty} \|du\|_{L^2} \|v\|_{L^2} \\ &\quad + (\|D\|_{L^\infty} + \lambda) \|u\|_{L^2} \|v\|_{L^2} + c_1 (|u + \omega_0|^{p-1} + 1) \|_{L^{p'}} \|v\|_{L^p}, \\ &\leq \|A\|_{L^\infty} \|du\|_{L^2} \|dv\|_{L^2} + c_3 \|B\|_{L^\infty} \|u\|_{L^p} \|dv\|_{L^2} + c_4 \|C\|_{L^\infty} \|du\|_{L^2} \|v\|_{L^p} \\ &\quad + c (\|D\|_{L^\infty} + \lambda) \|u\|_{L^p} \|v\|_{L^p} + (c_5 \|u\|_{L^p} + c_5 \|\omega_0\|_{L^p} - c_6) \|v\|_{L^p}, \end{aligned}$$

since Ω is bounded, i.e $|\Omega| < \infty$. So we need to check coercivity and monotonicity.

Coercivity

We begin by showing that there exists constants $\tilde{c}, \tilde{c}_1 > 0$ such that for all $u \in W_T^{d,2,p}(\Omega, \Lambda^k)$, we have,

$$\int_{\Omega} \langle \rho(x, u + \omega_0), u \rangle \geq \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1.$$

By (N2), we have,

$$\int_{\Omega} \langle \rho(x, u + \omega_0), u + \omega_0 \rangle \geq c_2 \|u + \omega_0\|_{L^p}^p - c_2 |\Omega|.$$

But this implies,

$$\int_{\Omega} \langle \rho(x, u + \omega_0), u \rangle \geq c_2 \|u + \omega_0\|_{L^p}^p - c_2 |\Omega| - \int_{\Omega} \langle \rho(x, u + \omega_0), \omega_0 \rangle.$$

Using Young's inequality with ε for the last term on the right, we deduce,

$$\begin{aligned} \int_{\Omega} \langle \rho(x, u + \omega_0), u \rangle &\geq c_2 \|u + \omega_0\|_{L^p}^p - c_2 |\Omega| - c_7 \varepsilon \int_{\Omega} |\rho(x, u + \omega_0)|^{p'} - \frac{1}{\varepsilon} c_8 \|\omega_0\|_{L^p}^p \\ &\geq c_2 \|u + \omega_0\|_{L^p}^p - c_9 \varepsilon \|u + \omega_0\|_{L^p}^p - c_{10}, \end{aligned}$$

where we have used (N1) in the last line again and c_{10} is a constant depending on ω_0, Ω and ε . Choosing ε small enough so that $c_2 - c_9 \varepsilon > 0$ yields,

$$\int_{\Omega} \langle \rho(x, u + \omega_0), u \rangle \geq c_{11} \|u + \omega_0\|_{L^p}^p - c_{12}.$$

This easily yields,

$$\int_{\Omega} \langle \rho(x, u + \omega_0), u \rangle \geq \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1.$$

On the other hand, since Ω is bounded, and $p \geq 2$, $W_T^{d,2,p}(\Omega, \Lambda^k) \subset W_T^{d,2}(\Omega, \Lambda^k)$. By Gårding's inequality, i.e by theorem 6.3, we know that there exists constants $\lambda_0 > 0$ and λ_1 such that,

$$a(u, u) \geq \lambda_0 \|du\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2,$$

for all $u \in W_T^{d,2,p}(\Omega, \Lambda^k)$. Hence for $\lambda > \lambda_1$, we have,

$$\begin{aligned} a_{\lambda,p}(u, u) &\geq \lambda_0 \|du\|_{L^2}^2 - \lambda_1 \|u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 + \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1 \\ &= \lambda_0 \|du\|_{L^2}^2 + (\lambda - \lambda_1) \|u\|_{L^2}^2 + \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1 \\ &\geq \lambda_0 \|du\|_{L^2}^2 + \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1 \\ &= \lambda_0 \left(\|du\|_{L^2}^2 + \|u\|_{L^p}^2 \right) + \tilde{c} \|u\|_{L^p}^p - \tilde{c}_1 - \lambda_0 \|u\|_{L^p}^2 \\ &= \lambda_0 \|u\|_{W^{d,2,p}}^2 + \tilde{c} \|u\|_{L^p}^p - \lambda_0 \|u\|_{L^p}^2 - \tilde{c}_1, \end{aligned}$$

for all $u \in W_T^{d,2,p}(\Omega, \Lambda^k)$. This means, we have, for all $\lambda \geq \lambda_1$,

$$a_{\lambda,p}(u, u) \geq c(\|u\|_{W^{d,2,p}}) \|u\|_{W^{d,2,p}},$$

for all $u \in W_T^{d,2,p}(\Omega, \Lambda^k)$, where

$$c(\|u\|_{W^{d,2,p}}) = \frac{\lambda_0 \|u\|_{W^{d,2,p}}^2 + \tilde{c} \|u\|_{L^p}^p - \lambda_0 \|u\|_{L^p}^2 - \tilde{c}_1}{\|u\|_{W^{d,2,p}}}.$$

Now we need to show $c(\|u\|_{W^{d,2,p}}) \rightarrow \infty$ when $\|u\|_{W^{d,2,p}} \rightarrow \infty$. But we have,

$$c(\|u\|_{W^{d,2,p}}) = \lambda_0 \|u\|_{W^{d,2,p}} + \frac{\tilde{c} \|u\|_{L^p}^p - \lambda_0 \|u\|_{L^p}^2 - \tilde{c}_1}{\|u\|_{W^{d,2,p}}}.$$

This implies $c(\|u\|_{W^{d,2,p}}) \rightarrow \infty$ when $\|u\|_{W^{d,2,p}} \rightarrow \infty$, since the second term on the right above is bounded below as $p \geq 2$. This proves coercivity.

Monotonicity

To prove monotonicity of the operator $a_{\lambda,p}$ we need to show,

$$a_{\lambda,p}(u, u - v) - a_{\lambda,p}(v, u - v) \geq 0 \quad \text{for all } u, v \in W_T^{d,2,p}(\Omega, \Lambda^k).$$

But

$$\begin{aligned} & a_{\lambda,p}(u, u - v) - a_{\lambda,p}(v, u - v) \\ &= a(u - v, u - v) + \lambda \int_{\Omega} \langle u - v, u - v \rangle + \int_{\Omega} (\langle \rho(x, u + \omega_0), u - v \rangle - \langle \rho(x, v + \omega_0), u - v \rangle) \\ &= a(u - v, u - v) + \lambda \|u - v\|_{L^2}^2 + \int_{\Omega} (\langle \rho(x, u + \omega_0), u - v \rangle - \langle \rho(x, v + \omega_0), u - v \rangle) \\ &\geq \lambda_0 \|d(u - v)\|_{L^2}^2 + (\lambda - \lambda_1) \|u - v\|_{L^2}^2 + \int_{\Omega} \langle \rho(x, u + \omega_0) - \rho(x, v + \omega_0), (u + \omega_0) - (v + \omega_0) \rangle, \end{aligned}$$

where we have used theorem 6.3 in the last inequality.

Combining (N3) and the last inequality above yields, for $\lambda \geq \lambda_1$,

$$a_{\lambda,p}(u, u - v) - a_{\lambda,p}(v, u - v) \geq 0.$$

This proves monotonicity.

Existence Setting $\tilde{\lambda} = \lambda_1$, we have shown that for $\lambda \geq \tilde{\lambda}$, the function $a_{\lambda,p} : W_T^{d,2,p}(\Omega, \Lambda^k) \times W_T^{d,2,p}(\Omega, \Lambda^k) \rightarrow \mathbb{R}$ is monotone and coercive on the reflexive Banach space $W_T^{d,2,p}(\Omega, \Lambda^k)$. Since for any $F \in L^2(\Omega, \Lambda^{k+1})$ and any $f \in L^{p'}(\Omega, \Lambda^k)$, where p' is the Hölder conjugate exponent of p , the map $\theta \mapsto -\int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle - a(\omega_0, \theta) - \lambda \int_{\Omega} \langle \omega_0, \theta \rangle$ defines a continuous linear functional on $W_T^{d,2,p}(\Omega, \Lambda^k)$, by theorem 3 in [18], we obtain the existence of $\tilde{\omega} \in W_T^{d,2,p}(\Omega, \Lambda^k)$ such that,

$$a_{\lambda,p}(\tilde{\omega}, \theta) = -\int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle - a(\omega_0, \theta) - \lambda \int_{\Omega} \langle \omega_0, \theta \rangle \quad \text{for all } \theta \in W_T^{d,2,p}(\Omega, \Lambda^k).$$

But this implies, for all $\theta \in W_T^{d,2,p}(\Omega, \Lambda^k)$,

$$a(\tilde{\omega} + \omega_0, \theta) + \lambda \int_{\Omega} \langle \tilde{\omega} + \omega_0, \theta \rangle + \int_{\Omega} \langle \rho(x, \tilde{\omega} + \omega_0), \theta \rangle + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0.$$

Setting $\omega = \tilde{\omega} + \omega_0$ completes the proof. ■

7.3 Semilinear theory: Strongly Indefinite case

For the class of semilinear equations we are going to discuss below, the energy functional is neither bounded above nor below. So we are going to look for the critical points of the energy functional instead. The strong indefiniteness of the functional will prevent direct use of standard critical point theory. We start with some abstract critical point theory that we can apply to such cases. We more or less follow Bartsch-Mederski [13] and Szulkin-Weth [66] with some modifications. The only real modification is basically to allow for a more general form of the linear operator. In Bartsch-Mederski, the term depending on derivatives of u of the functional $J(u)$ was $\frac{1}{2} \int_{\Omega} |du|^2$, whereas our modification allows for the term $\frac{1}{2} \int_{\Omega} \langle B(x) du, du \rangle$, where B is a bounded symmetric matrix field satisfying the Legendre-Hadamard condition in Ω .

7.3.1 Abstract critical point theory for indefinite functionals

Let X be a reflexive Banach space with a topological direct sum decomposition $X = X^+ \oplus \tilde{X}$. We also assume that the norm square is a C^1 map on X^+ , i.e the map which sends every $u \in X^+$ to $\|u\|^2 \in \mathbb{R}$ is $C^1(X^+; \mathbb{R})$ and hence, the intersection of the unit sphere with X^+ is a C^1 -submanifold of X^+ . Apart from the strong (norm) topology on X , we shall be using another topology on X . Let τ be the topology on X which is product of norm topology on X^+ and weak topology on \tilde{X} , i.e

$$u_n \xrightarrow{\tau} u, \quad \text{if and only if} \quad u_n^+ \rightarrow u^+ \quad \text{and} \quad \tilde{u}_n \rightharpoonup \tilde{u},$$

where $u_n = u_n^+ + \tilde{u}_n$ and $u = u^+ + \tilde{u}$, with the obvious meanings of the notations. For $u \in X \setminus \tilde{X}$, we define,

$$X(u) = \mathbb{R}u \oplus \tilde{X} \quad \text{and} \quad \hat{X}(u) = \mathbb{R}^+u \oplus \tilde{X}, \quad \text{where } \mathbb{R}^+ = [0, \infty).$$

Let $J \in C^1(X; \mathbb{R})$ be of the form,

$$J(u) = I^+(u^+) - I(u).$$

The following assumptions will be used throughout this section:

- (A1) $I \in C^1(X; \mathbb{R})$ and $I(u) \geq I(0) = 0$ for all $u \in X$.
- (A2) I is τ -sequentially lower semicontinuous.
- (A3) If $u_n \xrightarrow{\tau} u$ and $I(u_n) \rightarrow I(u)$, then $u_n \rightarrow u$.
- (A4) There exists $r > 0$ such that $a := \inf_{u \in X^+ : \|u\|=r} J(u) > 0$.
- (A5) For all $u \in X \setminus \tilde{X}$, there exists a unique critical point $0 \neq \hat{m}(u) \in \hat{X}(u)$ of $J|_{X(u)}$ and $\hat{m}(u)$ is the unique global maximum of $J|_{\hat{X}(u)}$.
- (A6) There exists $\delta > 0$ such that $\|\hat{m}(u)^+\| \geq \delta$ for all $u \in X \setminus \tilde{X}$ and \hat{m} is bounded on compact subsets of $X \setminus \tilde{X}$.

We now define the *Nehari-Pankov manifold* \mathcal{N} as the set

$$\mathcal{N} := \{\widehat{m}(u) : u \in X \setminus \widetilde{X}\}.$$

Theorem 7.3 *Let $J \in C^1(X; \mathbb{R})$ be of the form $J(u) = I^+(u^+) - I(u)$ and satisfy (A1)-(A6) and let $I^+ \in C^1(X; \mathbb{R})$ satisfy $I^+(0) = 0$. Let $c_0 := \inf_{\mathcal{N}} J$. Then the following holds:*

- (a) \mathcal{N} is homeomorphic to $S^+ := \{u \in X^+ : \|u\| = 1\}$ through the map $m := \widehat{m}|_{S^+}$.
- (b) $J \circ m \in C^1(S^+; \mathbb{R})$.
- (c) If $\{u_n\} \subset S^+$ is a Palais-Smale sequence for $J \circ m$, then $\{m(u_n)\} \subset \mathcal{N}$ is a Palais-Smale sequence for $J|_{\mathcal{N}}$. Conversely, if $\{m(u_n)\} \subset \mathcal{N}$ is a bounded Palais-Smale sequence for $J|_{\mathcal{N}}$, then $\{u_n\} \subset S^+$ is a Palais-Smale sequence for $J \circ m$ on S^+ .
- (d) $u \in S^+$ is a critical point of $J \circ m|_{S^+}$ if and only if $m(u) \in \mathcal{N}$ is a critical point of $J|_{\mathcal{N}}$.
- (e) J has a $(PS)_{c_0}$ sequence in \mathcal{N} .
- (f) If J satisfies $(PS)_{c_0}^\tau$ condition in \mathcal{N} , then c_0 is achieved by a critical point of J .

Proof (a) We shall show that the map m is a homeomorphism. Clearly, m is a bijection since by (A5), $m(u)$ is unique. The inverse $m^{-1} : \mathcal{N} \rightarrow S^+$ is given by,

$$m^{-1}(\widehat{m}(u)) = \frac{\widehat{m}(u^+)}{\|\widehat{m}(u^+)\|} \in S^+.$$

Note that since $\widehat{m}(u^+) \in \mathbb{R}^+ u$, $\frac{\widehat{m}(u^+)}{\|\widehat{m}(u^+)\|} = u$ for all $u \in S^+$.

Now we shall show that m is continuous. Suppose $\{w_i\} \subset X \setminus \widetilde{X}$, $w_i \rightarrow w \notin \widetilde{X}$. Since $\widehat{m}(w) = \widehat{m}\left(\frac{w^+}{\|w^+\|}\right)$, without loss of generality we can assume $w_i \in S^+$ for all $i \geq 1$. It is enough to show that $\widehat{m}(w_i) \rightarrow \widehat{m}(w)$ along a subsequence.

Let

$$\widehat{m}(w_i) = s_i w_i + v_i, \quad s_i \in \mathbb{R}^+, w_i \in S^+, v_i \in \widetilde{X} \text{ for all } i \geq 1.$$

Now since \widehat{m} is bounded on compact subsets, $\{\widehat{m}(w_i)\}$ is bounded. Hence, $\|s_i w_i\| \leq c$ and $\|v_i\| \leq c$ for some constant $c > 0$. But $\|s_i w_i\| = s_i$ since $w_i \in S^+$. This implies, along a subsequence,

$$s_i \rightarrow \bar{s} \quad \text{and} \quad v_i \rightharpoonup v^*,$$

for some $\bar{s} \in \mathbb{R}^+$ and some $v^* \in \widetilde{X}$.

This implies,

$$\widehat{m}(w_i) = s_i w_i + v_i \rightharpoonup \bar{s} w + v^* \quad \text{and} \quad s_i w_i \rightarrow \bar{s} w,$$

i.e $\widehat{m}(w_i) = s_i w_i + v_i \xrightarrow{\tau} \bar{s} w + v^*$. Let $\widehat{m}(w) = sw + v$.

So,

$$J(\widehat{m}(w_i)) = J(s_i w_i + v_i) \geq J(sw_i + v), \quad \text{since } \widehat{m}(w_i) \text{ is the unique maximum of } J|_{\widehat{X}(u)}.$$

Hence,

$$\lim_{i \rightarrow \infty} J(sw_i + v) \leq \liminf_{i \rightarrow \infty} J(\widehat{m}(w_i)).$$

But since $w_i \rightarrow w$ and $J \in C^1(X; \mathbb{R})$, $\lim_{i \rightarrow \infty} J(sw_i + v) = J(sw + v) = J(\widehat{m}(w))$. Hence,

$$\begin{aligned} J(\widehat{m}(w)) &\leq \liminf_{i \rightarrow \infty} J(\widehat{m}(w_i)) \\ &= \liminf_{i \rightarrow \infty} \{I^+(s_i w_i) - I(s_i w_i + v_i)\} \\ &\leq \lim_{i \rightarrow \infty} I^+(s_i w_i) - \liminf_{i \rightarrow \infty} I(s_i w_i + v_i) \\ &\leq I^+(\bar{s}w) - I(\bar{s}w + v^*) \quad (\text{by continuity of } I^+ \text{ and } \tau\text{-lower semicontinuity of } I) \\ &= J(\bar{s}w + v^*) \\ &\leq J(\widehat{m}(w)) \quad (\text{by maximality of } \widehat{m}(w)). \end{aligned}$$

Hence every inequality above must in fact, be equalities. Thus, $J(\widehat{m}(w)) = J(\bar{s}w + v^*)$. But then, by uniqueness of the maxima, we deduce,

$$v^* = v \quad \text{and } \bar{s} = s.$$

Hence,

$$\widehat{m}(w_i) \xrightarrow{\tau} \widehat{m}(w).$$

We also have,

$$\begin{aligned} J(\widehat{m}(w)) &= I^+(\bar{s}w) - \liminf_{i \rightarrow \infty} I(s_i w_i + v_i) \\ \Rightarrow I(\widehat{m}(w)) &= \liminf_{i \rightarrow \infty} I(s_i w_i + v_i). \end{aligned}$$

This implies $I(\widehat{m}(w_i)) \rightarrow I(\widehat{m}(w))$ along a subsequence. This together with the fact that $\widehat{m}(w_i) \xrightarrow{\tau} \widehat{m}(w)$ implies, by (A3),

$$\widehat{m}(w_i) \rightarrow \widehat{m}(w).$$

Hence, \widehat{m} , and thus m also, is continuous. It is easy to see that m^{-1} is continuous. So, $m : S^+ \rightarrow \mathcal{N}$ is a homeomorphism. This proves (a).

(b) We shall show that $J \circ m : S^+ \rightarrow \mathbb{R}$ is a C^1 map. Moreover, we shall also show that

$$(J \circ m)'(u) = \|m(u)^+\| J'(m(u))|_{T_u S^+} : T_u S^+ \rightarrow \mathbb{R} \text{ for all } u \in S^+, \quad (7.4)$$

where $T_u S^+$ is the tangent space of S^+ at the point $u \in S^+$. Note however that our hypotheses need not imply that m is a C^1 diffeomorphism. In the same vein, \mathcal{N} need not be a C^1 submanifold.

We define,

$$\widehat{\psi}(w) := J(\widehat{m}(w)) \quad \text{and } \psi := \widehat{\psi}|_{S^+}.$$

Let $w \in X^+ \setminus \{0\}$, $z \in X^+$, then, since $s_w w + v_w$ is the unique maximum of $J|_{\widehat{X}(w)}$,

$$\begin{aligned}\widehat{\psi}(w + tz) - \widehat{\psi}(w) &= J(s_{w+tz}(w + tz) + v_{w+tz}) - J(s_w w + v_w) \\ &\leq J(s_{w+tz}(w + tz) + v_{w+tz}) - J(s_{w+tz}w + v_{w+tz}), \\ &= J'(s_{w+tz}w + v_{w+tz} + \tau_t s_{w+tz}tz)s_{w+tz}tz,\end{aligned}$$

by mean value theorem, for all $|t|$ small enough and for some $\tau_t \in (0, 1)$. Hence,

$$\frac{\widehat{\psi}(w + tz) - \widehat{\psi}(w)}{t} \leq J'(s_{w+tz}w + v_{w+tz} + \tau_t s_{w+tz}tz)s_{w+tz}z.$$

Since m is continuous, the map $w \mapsto s_w$ is continuous since this is just the map $w \mapsto \|\widehat{m}(w)^+\|$ for $w \in S^+$. J' is also continuous since J is C^1 . This yields,

$$\limsup_{t \rightarrow 0} \frac{\widehat{\psi}(w + tz) - \widehat{\psi}(w)}{t} \leq J'(s_w w + v_w)s_w z.$$

Also, by similar arguments, since $s_{w+tz}(w + tz) + v_{w+tz}$ is the unique maximum of $J|_{\widehat{X}(w+tz)}$,

$$\begin{aligned}\widehat{\psi}(w + tz) - \widehat{\psi}(w) &= J(s_{w+tz}(w + tz) + v_{w+tz}) - J(s_w w + v_w) \\ &\geq J(s_w(w + tz) + v_w) - J(s_w w + v_w) \\ &= J'(s_w w + v_w + \eta_t s_w tz)s_w tz,\end{aligned}$$

by mean value theorem, for all $|t|$ small enough and for some $\eta_t \in (0, 1)$. The same continuity arguments as above yields,

$$\liminf_{t \rightarrow 0} \frac{\widehat{\psi}(w + tz) - \widehat{\psi}(w)}{t} \geq J'(s_w w + v_w)s_w z.$$

Hence, $\lim_{t \rightarrow 0} \frac{\widehat{\psi}(w + tz) - \widehat{\psi}(w)}{t}$ exists and

$$\widehat{\psi}'(w)z = \|m(w)^+\|J'(m(w))z \quad \text{for every } w \in S^+ \text{ and for every } z \in T_w S^+.$$

This proves (b) and also establishes (7.4).

(c) It is easy to see that we have the following decomposition,

$$X = T_u S^+ \oplus X(u).$$

Now, for all $u \in S^+$, we have,

$$\begin{aligned}
\|(J \circ m)'(u)\| &= \sup_{\substack{z \in T_u S^+ \\ \|z\|=1}} (J \circ m)'(u)z \\
&= \|m(u)^+\| \sup_{\substack{z \in T_u S^+ \\ \|z\|=1}} J'(m(u))z \\
&= \|m(u)^+\| \|J'(m(u))\|.
\end{aligned}$$

In the last line above, we have used the fact that

$$\|J'(m(u))\| = \sup_{\substack{v \in X \\ \|v\|=1}} J'(m(u))v = \sup_{\substack{z \in T_u S^+ \\ \|z\|=1}} J'(m(u))z,$$

since $J'(m(u))w = 0$ for all $w \in X(u)$, as $m(u)$ is a critical point of $J|_{X(u)}$.

Since $\|m(u)^+\|$ is uniformly bounded away from 0 for all $m(u) \in \mathcal{N}$, we deduce that there exists a constant $\delta > 0$ such that,

$$\|(J \circ m)'(u)\| \geq \delta \|J'(m(u))\|.$$

Hence,

$$(J \circ m)'(u_i) \rightarrow 0 \Rightarrow J'(m(u_i)) \rightarrow 0.$$

This proves that for every Palais-Smale sequence $\{u_i\} \subset S^+$ for $J \circ m$, $\{m(u_i)\} \subset \mathcal{N}$ is a Palais-Smale sequence for J .

Again, if $\{m(u_i)\} \subset \mathcal{N}$ is a bounded Palais-Smale sequence for J , there exists a constant $c > 0$ such that $\|m(u)^+\| \leq c$. Hence,

$$\|(J \circ m)'(u)\| \leq c \|J'(m(u))\|.$$

This yields,

$$J'(m(u_i)) \rightarrow 0 \Rightarrow (J \circ m)'(u_i) \rightarrow 0.$$

This completes the proof of (c).

(d) We showed already in the proof of (c) above that,

$$\|(J \circ m)'(u)\| = \|m(u)^+\| \|J'(m(u))\|.$$

As \mathcal{N} is bounded away from 0, $\|m(u)^+\|$ is always nonzero and (d) follows immediately. Moreover, the critical values are the same and $\inf_{S^+} J \circ m = \inf_{\mathcal{N}} J$. Also,

$$c_0 = \inf_{u \in \mathcal{N}} J(u) = \inf_{w \in X \setminus \tilde{X}} \max_{u \in \tilde{X}(w)} J(u) = \inf_{w \in S^+} \max_{u \in \tilde{X}(w)} J(u).$$

This last observation clearly shows that c_0 is actually a min-max value.

(e) Since $c_0 = \inf_{\mathcal{N}} J = \inf_{S^+} J \circ m$, there exists a minimizing sequence $\{v_i\}$ for $J \circ m$.

Also, since $J \circ m : S^+ \rightarrow \mathbb{R}$ is C^1 , we can apply Ekeland's variational principle to $\{v_i\}$ to obtain a sequence $\{w_i\} \subset S^+$ such that,

$$(J \circ m)(w_i) \rightarrow c_0 \quad \text{and} \quad (J \circ m)'(w_i) \rightarrow 0.$$

In other words, $\{w_i\} \subset S^+$ a $(PS)_{c_0}$ -sequence for $J \circ m$ and thus also a Palais-Smale sequence for $J \circ m$. Hence by (c), $\{m(w_i)\} \subset \mathcal{N}$ is a Palais-Smale sequence for J . But $(J \circ m)(w_i) \rightarrow c_0$ also implies that $\{m(w_i)\}$ is $(PS)_{c_0}$ -sequence for J on \mathcal{N} . This proves (e).

(f) We prove this in two steps.

Step 1 We first show that if J satisfies $(PS)_c^\tau$ -condition in \mathcal{N} for some $c > 0$, then $J \circ m$ satisfies $(PS)_c$ -condition on S^+ .

Consider a $(PS)_c$ -sequence $\{u_i\} \subset S^+$ for $J \circ m$. Then, by (c), $\{m(u_i)\}$ is a $(PS)_c$ -sequence for J on \mathcal{N} . If J satisfies the $(PS)_c^\tau$ -condition in \mathcal{N} , this implies that there exists $v \in X$ such that, along a subsequence,

$$m(u_i) \xrightarrow{\tau} v.$$

Note that we can not conclude yet that $v \in \mathcal{N}$ as \mathcal{N} need not be closed in τ -topology. However, \mathcal{N} is closed in the strong topology since it is homeomorphic to S^+ , which is closed under strong topology. Indeed, since X^+ , being a topologically complemented subspace, is closed and the norm is continuous on X^+ , S^+ is closed.

In particular, $m(u_i)^+ \rightarrow v^+$ and

$$\begin{aligned} 0 < c &= \lim_{i \rightarrow \infty} J(m(u_i)) \\ &= \lim_{i \rightarrow \infty} [I^+(m(u_i)^+) - I(m(u_i))] \\ &\leq \lim_{i \rightarrow \infty} I^+(m(u_i)^+) - \liminf_{i \rightarrow \infty} I(m(u_i)) \\ &\leq I^+(v^+) - I(v), \end{aligned}$$

by continuity of I^+ and τ -lower semicontinuity of I . Now since $I(v) \geq 0$, $I^+(v^+) - I(v) > 0$ implies $I^+(v^+) > 0$, which in turn implies $v^+ \neq 0$, since $I^+(0) = 0$. Hence $m(u_i)^+ \neq 0$ for i sufficiently large. Since $u_i = \frac{m(u_i)^+}{\|m(u_i)^+\|}$, we have,

$$u_i = \frac{m(u_i)^+}{\|m(u_i)^+\|} \rightarrow \frac{v^+}{\|v^+\|} \in S^+.$$

This proves that $J \circ m$ satisfies $(PS)_c$ -condition on S^+ .

Step 2 Now we complete the proof of (f).

By the proof of (e), we know there exists a $(PS)_{c_0}$ -sequence for $J \circ m$ in S^+ . Let $\{u_i\}$ be such a sequence. Since by step 1, $J \circ m$ satisfies $(PS)_c$ -condition on S^+ , there exist $u \in X$ such that $u_i \rightarrow u$ along a subsequence. Observe that $u \in S^+$ since S^+ is closed. Also, since $J \circ m$ is C^1 ,

this implies,

$$(J \circ m)(u_i) \rightarrow (J \circ m)(u) \quad \text{and} \quad (J \circ m)'(u_i) \rightarrow (J \circ m)'(u).$$

But $\{u_i\}$ is a $(PS)_{c_0}$ -sequence which implies,

$$(J \circ m)(u_i) \rightarrow c_0 \quad \text{and} \quad (J \circ m)'(u_i) \rightarrow 0.$$

Hence, we must have,

$$(J \circ m)(u) = c_0 \quad \text{and} \quad (J \circ m)'(u) = 0.$$

Thus u is a critical point for $J \circ m$ on S^+ . This implies, by (d), that $m(u)$ is a critical point for J in \mathcal{N} and $J(m(u)) = c_0$. This proves (f). ■

For verification of the hypothesis of the previous theorem, we introduce the following conditions:

(B1) $I^+(u^+) + I(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

(B2) $I(t_i u_i)/t_i^2 \rightarrow \infty$ if $t_i \rightarrow \infty$ and $u_i^+ \rightarrow u^+$ for some $u^+ \neq 0$, as $i \rightarrow \infty$.

(B3) $\frac{t^2-1}{2}I'(u)[u] + tI'(u)[v] + I(u) - I(tu + v) < 0$ for every $u \in X$, for every $t \geq 0$ and for every $v \in \tilde{X}$ with $u \neq tu + v$.

Proposition 7.4 *Let $J \in C^1(X; \mathbb{R})$ be of the form $J(u) = I^+(u^+) - I(u)$ and satisfy (A1)-(A2), (A4) and (B1)-(B3). Let $I^+ \in C^1(X; \mathbb{R})$ be of the form $I^+(u^+) := \frac{1}{2}B(u^+, u^+)$ where $B : X^+ \times X^+ \rightarrow \mathbb{R}$ is a symmetric continuous bilinear form. Then J satisfies (A5) and (A6).*

Proof Let $u \in X \setminus \tilde{X}$ and let $t_i u + \tilde{u}_i \rightarrow t_0 u + \tilde{u}_0$, where $\tilde{u}_i \in \tilde{X}$ and $t_i \geq 0$ for every $i \geq 0$. Then $t_i \rightarrow t_0$ and by τ -sequentially lower semicontinuity of I and continuity of I^+ , we obtain,

$$\liminf_{i \rightarrow \infty} \{-J(t_i u + \tilde{u}_i)\} = \liminf_{i \rightarrow \infty} \{I(t_i u + \tilde{u}_i) - I^+(t_i u)\} \geq I(t_0 u + \tilde{u}_0) - I^+(t_0 u) = -J(t_0 u + \tilde{u}_0).$$

This shows $-J$ is sequentially weakly lower semicontinuous on $\hat{X}(u)$.

Now we are going to show that $-J$ is coercive on $\hat{X}(u)$, i.e for any $\{v_i\} \subset \hat{X}(u)$ such that $\|v_i\| \rightarrow \infty$, $-J(v_i) \rightarrow \infty$. Suppose there exist a sequence $\{v_i\} \subset \hat{X}(u)$ such that $\|v_i\| \rightarrow \infty$. Now if there exists a subsequence of this sequence (not relabeled) such that $v_i^+ = 0$ for all i , then this implies,

$$-J(v_i) = I(v_i) - I^+(v_i^+) = I(v_i) \rightarrow \infty,$$

since by (B1), $I^+(v_i^+) + I(v_i) \rightarrow \infty$. This shows we can assume $v_i^+ \neq 0$. But then, since $\{v_i\} \subset \hat{X}(u)$, we can write, for each i , $v_i = t_i u + t_i \tilde{w}_i$, for some $w_i \in \tilde{X}$ and for some $t_i > 0$. Now if there is a subsequence such that $t_i \rightarrow \infty$ along that subsequence, then, setting $u_i = u + \tilde{w}_i$ and using (B2), we have,

$$-J(v_i) = I(v_i) - I^+(v_i^+) = I(t_i u_i) - I^+(t_i u) = I(t_i u_i) - t_i^2 I^+(u) = t_i^2 \left(\frac{I(t_i u_i)}{t_i^2} - I^+(u) \right) \rightarrow \infty.$$

This leaves open only the possibility that $t_i \leq C$ for all i . But then, (B1) implies, $t_i^2 I^+(u) + I(t_i u_i) \rightarrow \infty$, which in turn implies $I(t_i u_i) \rightarrow \infty$ and we have,

$$-J(v_i) = I(t_i u_i) - t_i^2 I^+(u) \geq I(t_i u_i) - C^2 I^+(u) \rightarrow \infty.$$

Thus $-J$ is coercive and lower semicontinuous on $\widehat{X}(u)$ and hence there exists a global maximum $\widehat{m}(u) \in \widehat{X}(u)$ of $J|_{\widehat{X}(u)}$. By (A4), $J(\widehat{m}(u)) \geq a > 0$, since $\widehat{m}(u)$ is the maximum on $\widehat{X}(u)$. Hence, $\widehat{m}(u) \notin \widetilde{X}$. Hence $\widehat{m}(u)$ is a critical point of $J|_{\widehat{X}(u)}$. Now we prove the uniqueness. Let $u \in X \setminus \widetilde{X}$ be any critical point of $J|_{\widehat{X}(u)}$. Note that for any $t \geq 0$ and $v \in \widetilde{X}$ such that $tu + v \neq u$, we have,

$$\begin{aligned} J(tu + v) - J(u) &= I^+(tu + v) - I^+(u) + I(u) - I(tu + v) \\ &= \frac{t^2 - 1}{2} B(u, u) + I(u) - I(tu + v). \end{aligned}$$

Now since u is a critical point of $J|_{\widehat{X}(u)}$, we have,

$$J'(u)z = 0,$$

for every $z \in \widehat{X}(u)$. Choosing $z = \frac{t^2 - 1}{2}u + tv$, we obtain,

$$0 = B(u, z) - I'(u)z = \frac{t^2 - 1}{2} B(u, u) - I'(u) \left(\frac{t^2 - 1}{2}u + tv \right).$$

This implies,

$$\begin{aligned} J(tu + v) - J(u) &= \frac{t^2 - 1}{2} B(u, u) + I(u) - I(tu + v) \\ &= I'(u) \left(\frac{t^2 - 1}{2}u + tv \right) + I(u) - I(tu + v) \\ &= \frac{t^2 - 1}{2} I'(u)[u] + t I'(u)[v] + I(u) - I(tu + v) \\ &< 0 \quad \text{by (B3).} \end{aligned}$$

This proves uniqueness.

To show that (A6) holds, note that, for any $u \in X \setminus \widetilde{X}$,

$$0 < a \leq J(\widehat{m}(u)) = I^+(\widehat{m}(u)^+) - I(\widehat{m}(u)),$$

by (A4) and by maximality of $\widehat{m}(u)$. Now since $I^+(\widehat{m}(u)^+) = \frac{1}{2} B(\widehat{m}(u)^+, \widehat{m}(u)^+)$ and B is a continuous, there exists a positive constant $c > 0$ such that,

$$I^+(\widehat{m}(u)^+) = \frac{1}{2} B(\widehat{m}(u)^+, \widehat{m}(u)^+) \leq c \|\widehat{m}(u)^+\|^2.$$

Since $I(\widehat{m}(u)) \leq 0$, this yields,

$$a \leq I^+(\widehat{m}(u)^+) \leq c\|\widehat{m}(u)^+\|^2.$$

This inequality proves the first part of (A6) with $\delta = \sqrt{\frac{a}{c}} > 0$.

For the second part, let $K \subset X \setminus \widetilde{X}$ be a compact subset such that \widehat{m} is not bounded on K . This implies there exists a sequence $\{u_i\} \subset K$ such that $\|\widehat{m}(u_i)\| \rightarrow \infty$. Let us write $\widehat{m}(u_i) = t_i u_i^+ + v_i$, where $v_i \in \widetilde{X}$ for all i . Note that compactness of K implies, passing to a subsequence if necessary, $u_i^+ \rightarrow u_0^+ \neq 0$ for some u_0 . Then, $J(t_i u_i^+ + v_i) > 0$ for all i . But this implies,

$$I^+(t_i u_i^+) > I(t_i u_i^+ + v_i).$$

This together with (B1) implies,

$$t_i^2 I^+(u_i^+) = I^+(t_i u_i^+) \rightarrow \infty.$$

Since K is compact and I^+ is continuous, $I^+(u_i^+)$ is uniformly bounded, which implies, by virtue of the last inequality, that $t_i \rightarrow \infty$. But then (B2) implies,

$$J(t_i u_i^+ + v_i) = t_i^2 I^+(u_i^+) - I(t_i u_i^+ + v_i) = t_i^2 \left(I^+(u_i^+) - \frac{I(t_i(u_i^+ + v_i/t_i))}{t_i^2} \right) \rightarrow -\infty,$$

which is impossible. This completes the proof of the proposition. ■

7.3.2 Existence of weak solutions

Theorem 7.5 (Ground state for semilinear Maxwell equation) *Let $n \geq 2$, $1 \leq k \leq n-1$, $2 < p < \frac{2n}{n-2}$ if $n > 2$ and $2 < p < \infty$ if $n = 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set. Let $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ be symmetric for a.e. $x \in \Omega$ and satisfy either the Legendre-Hadamard condition and is uniformly continuous or satisfy the Legendre ellipticity condition and is bounded and measurable. Let $\lambda \leq 0$. Let $W : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ be a map such that,*

(N1) *$W : \Omega \times \Lambda^k \rightarrow \mathbb{R}$ is differentiable with respect to $\xi \in \Lambda^k$ and the map $\rho(x, \xi) := \nabla_\xi W(x, \xi)$ is a Carathéodory function.*

(N2) *$|\rho(x, \xi)| = o(|\xi|)$ as $\xi \rightarrow 0$ uniformly in $x \in \Omega$.*

(N3) *There exists a constant $c_1 > 0$ such that,*

$$|\rho(x, \xi)| \leq c_1 (1 + |\xi|^{p-1}) \quad \text{for a.e } x \in \Omega, \text{ for all } \xi \in \Lambda^k.$$

(N4) *There exists a constant $c_2 > 0$ such that,*

$$\frac{1}{2} \langle \rho(x, \xi), \xi \rangle \geq W(x, \xi) > c_2 |\xi|^p \quad \text{for a.e } x \in \Omega, \text{ for all } \xi \in \Lambda^k.$$

(N5) *$\xi \mapsto W(x, \xi)$ is convex for a.e $x \in \Omega$. Also, if λ is an eigenvalue of the linear operator*

$L\omega = \delta(A(x)d\omega)$ on $V = \{\omega \in W_T^{d,2}(\Omega; \Lambda^k) : \delta\omega = 0 \text{ in the sense of distributions}\}$, then $\xi \mapsto W(x, \xi)$ is strictly convex and if $\lambda = 0$, $\xi \mapsto W(x, \xi)$ is uniformly strictly convex.

(N6) If $\langle \rho(x, \xi_1), \xi_2 \rangle = \langle \rho(x, \xi_2), \xi_1 \rangle \neq 0$, then

$$W(x, \xi_1) - W(x, \xi_2) \leq \frac{\langle \rho(x, \xi_1), \xi_1 \rangle^2 - \langle \rho(x, \xi_1), \xi_2 \rangle^2}{2\langle \rho(x, \xi_1), \xi_1 \rangle} \quad \text{for a.e } x \in \Omega, \text{ for all } \xi_1, \xi_2 \in \Lambda^k.$$

If $W(x, \xi_1) \neq W(x, \xi_2)$, then strict inequality holds.

Then there exists a nontrivial solution $\omega \in W_T^{d,2,p}(\Omega, \Lambda^k)$ to the following integro-differential equation,

$$\int_{\Omega} \langle A(x)d\omega, d\theta \rangle + \lambda \int_{\Omega} \langle \omega, \theta \rangle - \int_{\Omega} \langle \rho(x, \omega), \theta \rangle = 0,$$

for all $\theta \in W_T^{d,2,p}(\Omega, \Lambda^k)$. In other words, $\omega \in W^{d,2,p}(\Omega, \Lambda^k)$ is a nontrivial weak solution to the following boundary value problem,

$$\begin{cases} \delta(A(x)d\omega) + \rho(x, \omega) = \lambda\omega & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_0)$$

Remark 7.6 (i) As the proof will show, we actually prove the existence of a non-trivial ground state solution, i.e the existence of a nontrivial solution with minimum energy.

(ii) Note that here the sign of λ implies that we can solve the problem for λ in the direction of the spectrum of the linear operator L .

(iii) The hypotheses on the nonlinearity are satisfied, in particular, if $W(x, \xi) = V(x)|B(\xi)|^p$, with $V \in L^\infty(\Omega)$ and there is a constant $\alpha > 0$ such that $V(x) \geq \alpha > 0$ for a.e $x \in \Omega$, and $B : \Lambda^k \rightarrow \Lambda^k$ is an invertible linear map.

(iv) The hypotheses on the nonlinearity are obviously satisfied, in the special but somewhat prototypical case, when $W(x, \xi) = \frac{1}{p}|\xi|^p$, i.e $\rho(x, \omega) = |\omega|^{p-2}\omega$.

(iv) The above remark implies that the following boundary value problem,

$$\begin{cases} \delta(A(x)d\omega) + |\omega|^{p-2}\omega = \lambda\omega & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a ground-state solution $\omega \in W^{d,2,p}(\Omega, \Lambda^k)$ for every $\lambda \leq 0$.

Before proceeding with the proof of the theorem, we need several lemmas. We start by recalling the decomposition $W_T^{d,2,p}(\Omega; \Lambda^k) = V \oplus dW_0^{1,p}(\Omega; \Lambda^{k-1})$, where $V = W_{\delta,T}^{d,2}(\Omega; \Lambda^k)$. Also, since $A : \Omega \rightarrow L(\Lambda^{k+1}, \Lambda^{k+1})$ satisfies either the Legendre-Hadamard condition and is uniformly continuous or the Legendre ellipticity condition and is bounded and measurable, the linear operator $L\omega = \delta(A(x)d\omega)$ has a discrete non-increasing sequence of eigenvalues $\{\sigma_i\}_{i=1}^\infty$, each with finite multiplicity and each with a finite dimensional eigenspace in V by theorem 6.10. Let $v_i \in V$ be the eigenfunction corresponding to the eigenvalue σ_i , chosen such a way that $\{v_i\}_{i=1}^\infty$ is an orthonormal basis of V , which are orthogonal with respect to the inner products on $L^2(\Omega, \Lambda^k)$ and $W^{d,2}(\Omega, \Lambda^k)$.

We define,

$$n_0 := \min\{i \in \mathbb{N} : \lambda - \sigma_{i+1} > 0\} = \max\{i \in \mathbb{N} : \lambda - \sigma_i \leq 0\},$$

the dimension of the semi-negative eigenspace of the quadratic form,

$$Q(v) := \frac{1}{2} \int_{\Omega} \langle A(x)dv, dv \rangle + \frac{\lambda}{2} \int_{\Omega} |v|^2.$$

Let

$$V^+ := \text{span}\{v_i : i > n_0\} \quad \text{and} \quad \tilde{V} := \text{span}\{v_1, \dots, v_{n_0}\}.$$

For any $v \in V$, we write $v = v^+ + \tilde{v}$, where $v \in V^+$ and $\tilde{v} \in \tilde{V}$. Note that for any $v \in V^+$, there exists a constant $c > 0$ such that,

$$\frac{1}{2} \int_{\Omega} \langle A(x)dv, dv \rangle + \frac{\lambda}{2} \int_{\Omega} |v|^2 \geq c \int_{\Omega} |dv|^2 \quad \text{for all } v \in V^+. \quad (7.5)$$

Also,

$$\frac{1}{2} \int_{\Omega} \langle A(x)dv, dv \rangle + \frac{\lambda}{2} \int_{\Omega} |v|^2 \leq 0 \quad \text{for all } v \in \tilde{V}. \quad (7.6)$$

Let us set $X = V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$ and also,

$$X^+ := \{(v, 0) : v \in V^+\} \subset V^+ \times \{0\} \subset X$$

and

$$\tilde{X} := \{(v, w) : v \in \tilde{V}, w \in W_0^{1,p}(\Omega; \Lambda^{k-1})\} \subset \tilde{V} \times W_0^{1,p}(\Omega; \Lambda^{k-1}) \subset X.$$

We consider the functional $J : X \rightarrow \mathbb{R}$ defined by,

$$J(v, w) = \frac{1}{2} \int_{\Omega} \langle A(x)dv, dv \rangle + \frac{\lambda}{2} \int_{\Omega} |v + dw|^2 - \int_{\Omega} W(x, v + dw).$$

Note that this functional has the form $J((v, w)) = I^+(v) - I((v, w))$ with

$$I^+(v) = \frac{1}{2} \int_{\Omega} \langle A(x)dv^+, dv^+ \rangle + \frac{\lambda}{2} \int_{\Omega} |v^+|^2$$

and

$$I((v, w)) = - \left(\frac{1}{2} \int_{\Omega} \langle A(x)d\tilde{v}, d\tilde{v} \rangle + \frac{\lambda}{2} \int_{\Omega} |\tilde{v}|^2 \right) - \frac{\lambda}{2} \int_{\Omega} |dw|^2 + \int_{\Omega} W(x, v + dw).$$

Lemma 7.7 *The hypothesis on the nonlinearity implies, for every $\varepsilon > 0$, there is a constant C_{ε} such that,*

$$|\rho(x, \xi)| \leq \varepsilon |\xi| + C_{\varepsilon} |\xi|^{p-1} \quad \text{for any } \xi \in \Lambda^k, \text{ for a.e. } x \in \Omega, \quad (7.7)$$

and

$$\int_{\Omega} W(x, u) \leq \varepsilon \|u\|_{L^2}^2 + C_{\varepsilon} \|u\|_{L^p}^p \quad \text{for any } u \in L^p(\Omega; \Lambda^k). \quad (7.8)$$

Proof Indeed if the estimate (7.7) is false, this implies for every $n \in \mathbb{N}$, there exists $x_n \in \Omega$, $\xi_n \in \Lambda^k$ such that,

$$|\rho(x_n, \xi_n)| > \varepsilon |\xi_n| + n |\xi_n|^{p-1}.$$

But, then (N3) implies,

$$\begin{aligned} c_1 (1 + |\xi|^{p-1}) &> |\rho(x_n, \xi_n)| > \varepsilon |\xi_n| + n |\xi_n|^{p-1} \\ \Rightarrow c_1 &> \varepsilon |\xi_n| + (n - c_1) |\xi_n|^{p-1}. \end{aligned}$$

This implies, $\{\xi_n\}$ is bounded. Passing to a subsequence if necessary, we may assume, $\xi_n \rightarrow \xi \in \Lambda^k$. If $\xi \neq 0$, then this means, $|\xi_n|$ is bounded away from 0 for large n . This implies $\rho(x_n, \xi_n) \rightarrow \infty$, but this is a contradiction since $c_1 (1 + |\xi_n|^{p-1}) > |\rho(x_n, \xi_n)|$ and the left hand side is bounded. But if $\xi = 0$, then by (N2), there exists an integer N such that

$$\frac{|\rho(x, \xi_n)|}{|\xi_n|} < \varepsilon \quad \text{for all } n \geq N.$$

This implies,

$$0 > \frac{|\rho(x, \xi_N)|}{|\xi_N|} - \varepsilon > N |\xi_N|^{p-2},$$

which is impossible since the last term on the right is clearly nonnegative. This proves (7.7). Using this and integrating, we obtain the estimate (7.8). ■

Lemma 7.8 *The hypothesis on the nonlinearity implies,*

- (a) *I is of class C^1 , $I((v, w)) \geq 0$ for any $(v, w) \in V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$ and I is τ -sequentially lower semicontinuous.*
- (b) *There exists $r > 0$ such that $0 < \inf_{\substack{v \in V^+ \\ \|v\|_V = r}} J(v, 0)$.*
- (c) *$I^+(v^+) + I((v, w)) \rightarrow \infty$ as $\|(v, w)\| \rightarrow \infty$.*
- (d) *$I(t_i(v_i, w_i))/t_i^2 \rightarrow \infty$ if $t_i \rightarrow \infty$ and $v_i^+ \rightarrow v_0^+$ for some $v_0^+ \neq 0$, as $i \rightarrow \infty$.*

Proof (a) By (7.6) and since $\lambda \leq 0$, we see,

$$I((v, w)) = - \left(\frac{1}{2} \int_{\Omega} \langle A(x) d\tilde{v}, d\tilde{v} \rangle + \frac{\lambda}{2} \int_{\Omega} |\tilde{v}|^2 \right) - \frac{\lambda}{2} \int_{\Omega} |dw|^2 + \int_{\Omega} W(x, v + dw) \geq 0.$$

Now if $(v_i, w_i) \xrightarrow{\tau} (v_0, w_0)$, then we can assume $\tilde{v}_i \rightarrow \tilde{u}_0$ in V , since \tilde{V} is finite-dimensional. Then, since $\xi \mapsto W(x, \xi)$ is convex and hence $\int_{\Omega} W(x, v + dw)$ is sequentially weakly lower semi continuous, we deduce the the sequential τ -lower semicontinuity of I .

(b) For every $v \in V^+$, by (7.8) and (7.5), the embedding of V in L^2 and L^p and choosing ε small enough, we have, for some constant $C_1 > 0$,

$$\begin{aligned} J(v, 0) &= \frac{1}{2} \int_{\Omega} \langle A(x) dv, dv \rangle + \frac{\lambda}{2} \int_{\Omega} |v|^2 - \int_{\Omega} W(x, v) \\ &\geq c \int_{\Omega} |dv|^2 - \varepsilon \|v\|_{L^2}^2 - C_{\varepsilon} \|v\|_{L^p}^p \\ &\geq \frac{c}{2} \|v\|_V^2 - C_1 \|v\|_V^p. \end{aligned}$$

This implies (b), since $p > 2$.

(c) Note that by (7.5), $I^+(v^+) \geq c \|v^+\|_V^2$. Hence if $\|v^+\|_V \rightarrow \infty$, then $I^+(v^+) + I((v, w)) \rightarrow \infty$, since $I((v, w)) \geq 0$. Thus we suppose now that $\|(v_i, w_i)\| \rightarrow \infty$ with $\|v_i^+\|_V$ uniformly bounded. This means that $\|v_i + dw_i\|_{L^p} \rightarrow \infty$. This implies,

$$\begin{aligned} I((v_i, w_i)) &= - \left(\frac{1}{2} \int_{\Omega} \langle A(x) d\tilde{v}_i, d\tilde{v}_i \rangle + \frac{\lambda}{2} \int_{\Omega} |\tilde{v}_i|^2 \right) - \frac{\lambda}{2} \int_{\Omega} |dw_i|^2 + \int_{\Omega} W(x, v_i + dw_i) \\ &\geq \int_{\Omega} W(x, v_i + dw_i) \\ &\geq c_2 \|v_i + dw_i\|_{L^p}^p \rightarrow \infty, \end{aligned}$$

where we have used (N4) in the penultimate step.

(d) We have,

$$\begin{aligned} I(t_i(v_i, w_i)) &= - \left(\frac{t_i^2}{2} \int_{\Omega} \langle A(x) d\tilde{v}_i, d\tilde{v}_i \rangle + \frac{\lambda t_i^2}{2} \int_{\Omega} |\tilde{v}_i|^2 \right) - \frac{\lambda t_i^2}{2} \int_{\Omega} |dw_i|^2 + \int_{\Omega} W(x, t_i(v_i + dw_i)) \\ &\geq \int_{\Omega} W(x, t_i(v_i + dw_i)) \\ &\geq c_2 t_i^p \|v_i + dw_i\|_{L^p}^p. \end{aligned}$$

This implies,

$$I(t_i(v_i, w_i))/t_i^2 \geq c_2 t_i^{p-2} \|v_i + dw_i\|_{L^p}^p.$$

Since $p > 2$, this implies (d). Indeed, this conclusion can only fail if $\|v_i + dw_i\|_{L^p} \rightarrow 0$, which implies $\|v_i + dw_i\|_{L^2} \rightarrow 0$, which in turn implies, by orthogonality, $\|v_i\|_{L^2} \rightarrow 0$, an impossibility since $v_i^+ \rightarrow v_0^+ \neq 0$. This completes the proof. ■

Next we show that condition (B3) holds, which is the content of the next lemma.

Lemma 7.9 *For every $(v, w) \in V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$, for every $t \geq 0$ and for every $\phi \in \tilde{V}, \psi \in W_0^{1,p}(\Omega; \Lambda^{k-1})$. such that $v + dw \neq t(v + dw) + \phi + d\psi$,*

$$\frac{t^2 - 1}{2} I'((v, w))[(v, w)] + t I'((v, w))[(\phi, \psi)] + I((v, w)) - I(t(v, w) + (\phi, \psi)) < 0.$$

Proof We have, by a simple calculation,

$$\begin{aligned} & \frac{t^2 - 1}{2} I'((v, w))[(v, w)] + t I'((v, w))[(\phi, \psi)] + I((v, w)) - I(t(v, w) + (\phi, \psi)) \\ &= \frac{1}{2} \int_{\Omega} \langle A(x) d\phi, d\phi \rangle + \frac{\lambda}{2} \int_{\Omega} |\phi|^2 + \frac{\lambda}{2} \int_{\Omega} |d\psi|^2 + \int_{\Omega} \Phi(t, x), \end{aligned}$$

where

$$\Phi(t, x) := \langle \rho(x, v + dw), \frac{t^2 - 1}{2}(v + dw) + t(\phi + d\psi) \rangle + W(x, v + dw) - W(x, t(v + dw) + \phi + d\psi).$$

We shall show,

$$\frac{1}{2} \int_{\Omega} \langle A(x) d\phi, d\phi \rangle + \frac{\lambda}{2} \int_{\Omega} |\phi|^2 + \frac{\lambda}{2} \int_{\Omega} |d\psi|^2 + \int_{\Omega} \Phi(t, x) < 0. \quad (7.9)$$

Note that by (N4), we have, $\Phi(0, x) < 0$. Note also that since among the terms containing t , the one with W grows like p -th power whereas the terms involving ρ grows at most quadratically, we have,

$$\lim_{t \rightarrow \infty} \Phi(t, x) = -\infty.$$

Hence, $t \mapsto \Phi(t, x)$ achieves a maximum on $[0, \infty)$ for some $t \geq 0$. Let $t_0 \geq 0$ be such that $\Phi(t_0, x) = \max_{t \geq 0} \Phi(t, x)$. If $t_0 = 0$, then $\Phi(t, x) < 0$ for all $t \geq 0$. So we can assume $t_0 > 0$. Then

$$\left. \frac{\partial \Phi(t, x)}{\partial t} \right|_{t=t_0} = 0.$$

This implies,

$$\langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle - \langle \rho(x, t_0(v + dw) + \phi + d\psi), v + dw \rangle = 0.$$

If $\langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle = 0$, then by (N4),

$$\begin{aligned} & \Phi(t_0, x) \\ &= \langle \rho(x, v + dw), \frac{t_0^2 - 1}{2}(v + dw) + t_0(\phi + d\psi) \rangle + W(x, v + dw) - W(x, t_0(v + w) + \phi + d\psi) \\ &= \langle \rho(x, v + dw), \frac{-t_0^2 - 1}{2}(v + dw) + t_0(\phi + d\psi) \rangle + t_0 \langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle \\ & \quad + W(x, v + dw) - W(x, t_0(v + w) + \phi + d\psi) \\ &= \langle \rho(x, v + dw), \frac{-t_0^2 - 1}{2}(v + dw) + t_0(\phi + d\psi) \rangle + W(x, v + dw) - W(x, t_0(v + w) + \phi + d\psi) \\ &\leq -t_0^2 W(x, v + dw) - W(x, t_0(v + dw) + \phi + d\psi) \\ &< 0. \end{aligned}$$

If $\langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle = \langle \rho(x, t_0(v + dw) + \phi + d\psi), v + dw \rangle \neq 0$, then by (N6),

$$\begin{aligned}
& \Phi(t_0, x) \\
&= -\frac{(t_0 - 1)^2}{2} \langle \rho(x, v + dw), v + dw \rangle + t_0 \langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle \\
&\quad - t_0 \langle \rho(x, v + dw), v + dw \rangle + W(x, v + dw) - W(x, t_0(v + dw) + \phi + d\psi) \\
&\leq -\frac{\langle \rho(x, v + dw), \phi + d\psi \rangle^2}{2 \langle \rho(x, v + dw), v + dw \rangle} \\
&\leq 0.
\end{aligned}$$

If $W(x, v + dw) \neq W(x, t_0(v + dw) + \phi + d\psi)$, then $\Phi(t_0, x) < 0$ and if $W(x, v + dw) = W(x, t_0(v + dw) + \phi + d\psi)$, then again (N6) implies,

$$\langle \rho(x, v + dw), t_0(v + dw) + \phi + d\psi \rangle \leq \langle \rho(x, v + dw), v + dw \rangle.$$

This implies, as above,

$$\Phi(t_0, x) \leq -\frac{(t_0 - 1)^2}{2} \langle \rho(x, v + dw), v + dw \rangle \leq 0,$$

and the inequality is strict if $t_0 \neq 1$. Finally, if $t_0 = 1$ and there exists $0 < t \neq t_0$ such that $\Phi(t_0, x) = \Phi(t, x)$, then the above discussion yields $\Phi(t, x) < 0$ for all $t \geq 0$. Thus we have shown that $\Phi(t, x) < 0$ for all $t \geq 0, t \neq 1$. This implies (7.9) if $t \neq 1$. Now for the case $t = 1$, if $\lambda \neq 0$ and λ is not an eigenvalue of linear operator $L\omega = \delta(A(x)d\omega)$ on V , then we have,

$$\frac{1}{2} \int_{\Omega} \langle A(x)d\phi, d\phi \rangle + \frac{\lambda}{2} \int_{\Omega} |\phi|^2 + \frac{\lambda}{2} \int_{\Omega} |d\psi|^2 < 0,$$

proving the result. Otherwise, by (N5), $\xi \mapsto W(x, \xi)$ is strictly convex and this implies,

$$\Phi(1, x) = \langle \rho(x, v + dw), \phi + d\psi \rangle + W(x, v + dw) - W(x, v + dw + \phi + d\psi) < 0.$$

This proves the result. ■

Next we define the Nehari-Pankov manifold \mathcal{N} for J as,

$$\begin{aligned}
\mathcal{N} := \Big\{ (v, w) \in V \times W_0^{1,p}(\Omega; \Lambda^{k-1}) \setminus \widetilde{V} \times W_0^{1,p}(\Omega; \Lambda^{k-1}) \Big| & J'(v, w)[v, w] = 0 \\
& \text{and } J'(v, w)[\phi, \psi] = 0 \text{ for any } (\phi, \psi) \in \widetilde{V} \times W_0^{1,p}(\Omega; \Lambda^{k-1}) \Big\}
\end{aligned}$$

We now show that J satisfies the $(PS)_c^T$ condition on \mathcal{N} for any $c > 0$.

Lemma 7.10 *If $\{(v_i, w_i)\} \subset \mathcal{N}$ is a $(PS)_c$ sequence for J on \mathcal{N} for some $c > 0$, i.e if*

$$J(v_i, w_i) \rightarrow c \quad \text{and} \quad J'(v_i, w_i) \rightarrow 0,$$

then, passing to a a subsequence which we do not relabel, we have,

$$(v_i, w_i) \xrightarrow{T} (v_0, w_0) \quad \text{for some } (v_0, w_0) \in V \times W_0^{1,p}(\Omega; \Lambda^{k-1}).$$

Proof First we show that such a sequence $\{(v_i, w_i)\}$ must be bounded. We argue by contradiction. Suppose $\|(v_i, w_i)\| \rightarrow \infty$ and we set,

$$\bar{v}_i := \frac{v_i}{\|(v_i, w_i)\|} \quad \text{and} \quad \bar{w}_i := \frac{w_i}{\|(v_i, w_i)\|}.$$

Since $\{\|\bar{v}_i\|_V\}$ is bounded, we can suppose, passing to a subsequence if necessary, that,

$$\bar{v}_i \rightharpoonup \bar{v}_0 \quad \text{in } V.$$

By compact embedding of V into $L^p(\Omega; \Lambda^k)$, this implies,

$$\bar{v}_i \rightarrow \bar{v}_0 \quad \text{in } L^p(\Omega; \Lambda^k).$$

This in turn implies,

$$\bar{v}_i(x) \rightarrow \bar{v}_0(x) \quad \text{for a.e. } x \in \Omega.$$

We first show that $\bar{v}_0 \neq 0$. Let $\bar{V}^{\|\cdot\|_{L^p}}$ denote the closure of V in $L^p(\Omega; \Lambda^k)$. Also $dW_0^{1,p}(\Omega; \Lambda^{k-1})$ is a closed subspace of $L^p(\Omega; \Lambda^k)$ and $\bar{V}^{\|\cdot\|_{L^p}} \cap dW_0^{1,p}(\Omega; \Lambda^{k-1}) = \{0\}$. Hence by using the continuity of the projection map from $\bar{V}^{\|\cdot\|_{L^p}} \oplus dW_0^{1,p}(\Omega; \Lambda^{k-1})$ onto $\bar{V}^{\|\cdot\|_{L^p}}$ and $dW_0^{1,p}(\Omega; \Lambda^{k-1})$ in $L^p(\Omega; \Lambda^k)$, we deduce there exists a constant $C_2 > 0$ such that,

$$\|v\|_{L^p} \leq C_2 \|v + dw\|_{L^p} \quad \text{for any } v \in V, \quad (7.10)$$

and

$$\|dw\|_{L^p} \leq C_2 \|v + dw\|_{L^p} \quad \text{for any } w \in W_0^{1,p}(\Omega; \Lambda^{k-1}). \quad (7.11)$$

Now, assumptions on A and (N4) implies, for some $C_3 > 0$, we have,

$$\begin{aligned} C_3 \|v_i\|_V^2 &\geq \frac{1}{2} \int_{\Omega} \langle A(x) dv_i, dv_i \rangle \\ &= J(v_i, w_i) - \frac{\lambda}{2} \int_{\Omega} |v_i + dw_i|^2 + \int_{\Omega} W(x, v_i + dw_i) \\ &\geq J(v_i, w_i) - \frac{\lambda}{2} \int_{\Omega} |v_i + dw_i|^2 + c_2 \|v_i + dw_i\|_{L^p}^p \\ &\geq J(v_i, w_i) + c_2 \|v_i + dw_i\|_{L^p}^p. \end{aligned}$$

As $J(v_i, w_i) \rightarrow c > 0$, this implies, for i large enough, we can have,

$$\begin{aligned} 2C_3 \|v_i\|_V^2 &\geq C_3 \|v_i\|_V^2 + \frac{c}{2} + c_2 \|v_i + dw_i\|_{L^p}^p \\ &\geq C_3 \|v_i\|_V^2 + \frac{c_2}{(C_2)^p} \|dw_i\|_{L^p}^p \\ &\geq C_3 \|v_i\|_V^2 + C_3 \|dw_i\|_{L^p}^2 \\ &\geq C_4 \|(v_i, w_i)\|^2. \end{aligned}$$

This implies that $\|\bar{v}_i\|_V$ is uniformly bounded away from 0. But by lemma 7.8, lemma 7.9 and

proposition 7.4, we have that J satisfies the condition (A5) and hence, we have,

$$J(v_i, w_i) \geq J(t\bar{v}_i^+, 0) \geq t^2 \left(\frac{1}{2} \int_{\Omega} \langle A(x) d\bar{v}_i^+, d\bar{v}_i^+ \rangle + \frac{\lambda}{2} \int_{\Omega} |\bar{v}_i^+|^2 \right) \geq c_0 t^2 \int_{\Omega} |d\bar{v}_i^+|^2,$$

for any $t \geq 0$. That is,

$$J(v_i, w_i) \geq c_0 t^2 \|\bar{v}_i^+\|_V^2.$$

Hence, we have,

$$c \geq c_0 t^2 \liminf_{n \rightarrow \infty} \|\bar{v}_i^+\|_V^2.$$

Letting $t \rightarrow +\infty$, we obtain, $\liminf_{n \rightarrow \infty} \|\bar{v}_i^+\|_V^2 = 0$. But if $\bar{v}_0 = 0$, then since \widetilde{V} is finite dimensional, this implies $\widetilde{\bar{v}_i^+} \rightarrow \widetilde{\bar{v}_0} = 0$, which means $\liminf_{n \rightarrow \infty} \|\bar{v}_i\|_V^2 = \liminf_{n \rightarrow \infty} \|\bar{v}_i^+\|_V^2 = 0$. This contradicts the fact that $\|\bar{v}_i\|_V$ is uniformly bounded away from 0 and proves $\bar{v}_0 \neq 0$.

Again, as before, we have,

$$J(v_i, w_i) \leq C_3 \|v_i\|_V^2 - c_2 \|v_i + dw_i\|_{L^p}^p.$$

This implies, by (7.10),

$$J(v_i, w_i) \leq C_3 \|v_i\|_V^2 - \frac{c_2}{C_2} \|v_i\|_{L^p}^p.$$

Dividing by $\|(v_i, w_i)\|^2$, we obtain,

$$\frac{J(v_i, w_i)}{\|(v_i, w_i)\|^2} \leq C_3 \|\bar{v}_i\|_V^2 - \frac{c_2}{C_2} \int_{\Omega} |v_i|^{p-2} |\bar{v}_i|^2 \rightarrow -\infty,$$

as $\int_{\Omega} |v_i|^{p-2} |\bar{v}_i|^2 \rightarrow \infty$, by Fatou's lemma, as $v_i = \bar{v}_i \|(v_i, w_i)\|^2 \rightarrow \infty$. But this contradicts the fact that $\frac{J(v_i, w_i)}{\|(v_i, w_i)\|^2} \rightarrow 0$ as $J(v_i, w_i) \rightarrow c > 0$ and $\|(v_i, w_i)\| \rightarrow \infty$. Hence, $\{(v_i, w_i)\}$ is a bounded sequence.

So we can assume that up to subsequence which we do not relabel, we have,

$$v_i \rightharpoonup v_0 \text{ in } V, \quad v_i \rightarrow v_0 \text{ in } L^p(\Omega; \Lambda^k), \quad w_i \rightharpoonup w_0 \text{ in } W_0^{1,p}(\Omega; \Lambda^{k-1}),$$

for some $(v_0, w_0) \in V \times dW_0^{1,p}(\Omega; \Lambda^{k-1})$. Now, by Gårding inequality (cf. theorem 6.3), we have,

$$\begin{aligned} & J'(v_i, w_i)[v_i - v_0, 0] \\ &= \int_{\Omega} \langle A(x) dv_i, dv_i - dv_0 \rangle + \int_{\Omega} \langle v_i + dw_i, v_i - v_0 \rangle - \int_{\Omega} \langle \rho(x, v_i + dw_i), v_i - v_0 \rangle \\ &= \int_{\Omega} \langle A(x) dv_i - dv_0, dv_i - dv_0 \rangle + \int_{\Omega} \langle A(x) dv_0, dv_i - dv_0 \rangle + \int_{\Omega} \langle v_i + dw_i, v_i - v_0 \rangle \\ & \quad - \int_{\Omega} \langle \rho(x, v_i + dw_i), v_i - v_0 \rangle \\ &\geq 2\lambda_0 \|v_i - v_0\|_V^2 - \lambda_1 \|v_i - v_0\|_{L^2}^2 + \int_{\Omega} \langle A(x) dv_0, dv_i - dv_0 \rangle + \int_{\Omega} \langle v_i + dw_i, v_i - v_0 \rangle \\ & \quad - \int_{\Omega} \langle \rho(x, v_i + dw_i), v_i - v_0 \rangle. \end{aligned}$$

Since $J'(v_i, w_i) \rightarrow 0$, $v_i \rightharpoonup v_0$ in V , which implies $dv_i \rightharpoonup dv_0$ in L^2 , $v_i \rightarrow v_0$ in L^2 and $\{\rho(x, v_i + dw_i)\}_i$ is uniformly bounded in $L^{\frac{p}{p-1}}$, we obtain $\|v_i - v_0\|_V \rightarrow 0$. This yields $(v_i, w_i) \xrightarrow{\tau} (v_0, w_0)$ and finishes the proof. ■

We need just one more lemma, which shows that the condition (A3) is satisfied.

Lemma 7.11 *Let $\{(v_i, w_i)\} \subset V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$ be a sequence such that $(v_i, w_i) \xrightarrow{\tau} (v_0, w_0)$ and $I((v_i, w_i)) \rightarrow I((v_0, w_0))$. Then, $(v_i, w_i) \rightarrow (v_0, w_0)$ in $V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$.*

Proof It is enough to prove the result up to passing to a subsequence. Now, $(v_i, w_i) \xrightarrow{\tau} (v_0, w_0)$ implies, since \tilde{V} is finite dimensional,

$$\begin{aligned} v_i^+ \rightarrow v_0^+ \text{ in } V, \quad \tilde{v}_i \rightarrow \tilde{v}_0 \text{ in } V, \quad w_i \rightharpoonup w_0 \text{ in } W_0^{1,p}(\Omega; \Lambda^{k-1}) \\ \text{and } v_i + dw_i \rightharpoonup v_0 + dw_0 \text{ in } L^p(\Omega; \Lambda^k). \end{aligned}$$

Hence, $I((v_i, w_i)) \rightarrow I((v_0, w_0))$ implies,

$$-\frac{\lambda}{2} \int_{\Omega} |dw_i|^2 + \int_{\Omega} W(x, v_i + dw_i) \rightarrow -\frac{\lambda}{2} \int_{\Omega} |dw_0|^2 + \int_{\Omega} W(x, v_0 + dw_0).$$

Now if $\lambda < 0$, then we have, by sequential weak lower semicontinuity,

$$\begin{aligned} -\frac{\lambda}{2} \int_{\Omega} |dw_0|^2 + \int_{\Omega} W(x, v_0 + dw_0) &= -\frac{\lambda}{2} \lim_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2 + \lim_{i \rightarrow \infty} \int_{\Omega} W(x, v_i + dw_i) \\ &\geq -\frac{\lambda}{2} \lim_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2 + \liminf_{i \rightarrow \infty} \int_{\Omega} W(x, v_i + dw_i) \\ &\geq -\frac{\lambda}{2} \lim_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2 + \int_{\Omega} W(x, v_0 + dw_0). \end{aligned}$$

Hence, we have,

$$\int_{\Omega} |dw_0|^2 \geq \lim_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2.$$

But, again by sequential weak lower semicontinuity,

$$\lim_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2 \geq \liminf_{i \rightarrow \infty} \int_{\Omega} |dw_i|^2 \geq \int_{\Omega} |dw_0|^2.$$

This shows, $dw_i \rightarrow dw_0$ in L^2 , which yields, $dw_i \rightarrow dw_0$ a.e in Ω .

If $\lambda = 0$, then firstly, $I((v_i, w_i)) \rightarrow I((v_0, w_0))$ implies,

$$\int_{\Omega} W(x, v_i + dw_i) \rightarrow \int_{\Omega} W(x, v_0 + dw_0).$$

Now by (N5), i.e by uniform strict convexity of W , we have, for any $0 < r \leq R$,

$$m := \inf_{\substack{x \in \Omega, \xi_1, \xi_2 \in \Lambda^k, \\ |\xi_1 - \xi_2| \geq r, |\xi_1|, |\xi_2| \leq R}} \left\{ \frac{1}{2} [W(x, \xi_1) + W(x, \xi_2)] - W(x, \frac{\xi_1 + \xi_2}{2}) \right\} > 0.$$

Again, by convexity and sequential weak lower semicontinuity,

$$\begin{aligned}
0 &\leq \limsup_{i \rightarrow \infty} \left\{ \frac{1}{2} [W(x, v_i + dw_i) + W(x, v_0 + dw_0)] - W\left(x, \frac{v_i + dw_i + v_0 + dw_0}{2}\right) \right\} \\
&\leq \frac{1}{2} [W(x, v_0 + dw_0) + W(x, v_0 + dw_0)] - W\left(x, \frac{v_0 + dw_0 + v_0 + dw_0}{2}\right) \\
&= 0.
\end{aligned}$$

We set,

$$\Omega_i^{r,R} := \left\{ x \in \Omega, \left| v_i + dw_i - (v_0 + dw_0) \right| \geq r, |v_i + dw_i|, |v_0 + dw_0| \geq R \right\}.$$

Then we have,

$$m|\Omega_i^{r,R}| \leq \int_{\Omega} \left\{ \frac{1}{2} [W(x, v_i + dw_i) + W(x, v_0 + dw_0)] - W\left(x, \frac{v_i + dw_i + v_0 + dw_0}{2}\right) \right\}.$$

This implies, $|\Omega_i^{r,R}| \rightarrow 0$. Since $0 < r \leq R$ is arbitrary, we obtain, in this case also, $dw_i \rightarrow dw_0$ a.e in Ω .

Now we finish the proof of the lemma. We have, by what we have shown so far,

$$dw_i \rightarrow dw_0 \quad a.e. \quad \text{in } \Omega$$

and we want to show,

$$dw_i \rightarrow dw_0 \quad \text{in } L^p(\Omega; \Lambda^k).$$

By (N4), it will be enough to show,

$$\int_{\Omega} W(x, v_i + dw_i - (v_0 + dw_0)) \rightarrow 0.$$

But since $\int_{\Omega} W(x, v_i + dw_i) \rightarrow \int_{\Omega} W(x, v_0 + dw_0)$, this is equivalent to showing,

$$\int_{\Omega} \{W(x, v_i + dw_i) - W(x, v_i + dw_i - (v_0 + dw_0))\} \rightarrow \int_{\Omega} W(x, v_0 + dw_0).$$

Now, we have,

$$\begin{aligned}
&\int_{\Omega} \{W(x, v_i + dw_i) - W(x, v_i + dw_i - (v_0 + dw_0))\} dx \\
&= \int_{\Omega} \int_0^1 \frac{d}{dt} [W(x, v_i + dw_i + (t-1)(v_0 + dw_0))] dt dx \\
&= \int_0^1 \int_{\Omega} \langle \rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)), v_0 + dw_0 \rangle dx dt.
\end{aligned}$$

and

$$\int_{\Omega} W(x, v_0 + dw_0) = \int_0^1 \int_{\Omega} \langle \rho(x, t(v_0 + dw_0)), v_0 + dw_0 \rangle dx dt.$$

Since $v_i \rightarrow v_0$ in V , L^2 and L^p , we have $v_i \rightarrow v_0$ a.e in Ω up to a subsequence. Now $dw_i \rightarrow dw_0$ a.e in Ω implies $v_i + dw_i \rightarrow v_0 + dw_0$ a.e in Ω . This implies $\rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)) \rightarrow \rho(x, t(v_0 + dw_0))$ a.e in Ω , as ρ is Carathéodory, by Vitali convergence theorem, the lemma will be proved once we show that $\{\langle \rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)), v_0 + dw_0 \rangle\}_{i=1}^\infty$ is equiintegrable.

Therefore, we need to show, for every $\epsilon > 0$, there exists a $\delta > 0$ such that,

$$\int_E \langle \rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)), v_0 + dw_0 \rangle \leq \epsilon,$$

for all i and for any $E \subset \Omega$ with $|E| < \delta$.

But by (N3) and using Hölder inequality, we have,

$$\begin{aligned} & \int_E \langle \rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)), v_0 + dw_0 \rangle \\ & \leq c_1 \left(\int_E |v_0 + dw_0| + \int_E |v_i + dw_i + (t-1)(v_0 + dw_0)|^{p-1} |v_0 + dw_0| \right) \\ & \leq c_1 \left(\int_E |v_0 + dw_0| + \left(\int_E |v_i + dw_i + (t-1)(v_0 + dw_0)|^p \right)^{\frac{p-1}{p}} \left(\int_E |v_0 + dw_0|^p \right)^{\frac{1}{p}} \right) \\ & \leq c_1 \left(\int_E |v_0 + dw_0| + \|v_i + dw_i + (t-1)(v_0 + dw_0)\|_{L^p}^{p-1} \left(\int_E |v_0 + dw_0|^p \right)^{\frac{1}{p}} \right). \end{aligned}$$

Since $\|v_i + dw_i\|_{L^p}$ is uniformly bounded, as it is weakly convergent, we can rewrite the above inequalities as,

$$\int_E \langle \rho(x, v_i + dw_i + (t-1)(v_0 + dw_0)), v_0 + dw_0 \rangle \leq \tilde{C} \left(\int_E |v_0 + dw_0|^p \right)^{\frac{1}{p}},$$

for some constant $\tilde{C} > 0$.

Since $v_0 + dw_0 \in L^p(\Omega; \Lambda^k)$, we can find $\delta > 0$ such that,

$$\int_E |v_0 + dw_0|^p \leq \left(\epsilon / \tilde{C} \right)^p \text{ whenever } |E| < \delta.$$

This shows equiintegrability and finishes the proof of the lemma. ■

Now we are ready to finish the proof the theorem, which has been reduced to just a matter of stitching together the pieces by now.

Proof (*Theorem 7.5*) By the help of lemma 7.11, 7.8, 7.9 and proposition 7.4, we deduce that all the hypothesis of theorem 7.3 is satisfied. Hence by theorem 7.3 and lemma 7.10, we deduce that there exists a nontrivial critical point $(v, w) \in V \times W_0^{1,p}(\Omega; \Lambda^{k-1})$ of J such that $J((v, w)) = \inf_{\mathcal{N}} J$. Setting $\omega = v + dw \in V \oplus dW_0^{1,p}(\Omega; \Lambda^{k-1}) = W_T^{d,2,p}(\Omega, \Lambda^k)$, we obtain the theorem. ■

7.4 Quasilinear theory

7.4.1 Existence of weak solutions

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $1 < p < \infty$ and consider the following subspace $W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \subset W_T^{d,p}(\Omega; \Lambda^k)$ defined by,

$$W_{\delta,T}^{d,p}(\Omega; \Lambda^k) := \{\omega \in W_T^{d,p}(\Omega; \Lambda^k); \delta\omega = 0\},$$

where the condition $\delta\omega = 0$ is understood in the sense of distributions. Clearly $W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ is a closed subspace of $W_T^{d,p}(\Omega; \Lambda^k)$. Also, $dW_0^{1,p}(\Omega; \Lambda^k)$ is a closed subspace of $W_T^{d,p}(\Omega; \Lambda^k)$ and

$$W_T^{d,p}(\Omega; \Lambda^k) = W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \oplus dW_0^{1,p}(\Omega; \Lambda^k).$$

(cf. theorem 2.52 for the proof of the above decomposition and section 2.5 for related results). The direct sum decomposition is clearly also orthogonal with respect to the inner product. Also note that $W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ embeds continuously in $W^{1,p}$ and hence by Rellich's theorem, $W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ embeds compactly in L^p . Hence the norm $\|v\|_{W_{\delta,T}^{d,p}(\Omega; \Lambda^k)} = \|dv\|_{L^p}$ is an equivalent norm on $W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$.

Existence in $W_{\delta,T}^{d,p}$

Theorem 7.12 *Let $1 \leq k \leq n-1$ be an integer and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $A : \Omega \times \Lambda^{k+1} \rightarrow \Lambda^{k+1}$ be a measurable map such that*

(N1) *There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$|A(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) *There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$\langle A(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) *For every $u, v \in W^{d,p}(\Omega, \Lambda^k)$,*

$$\langle A(x, du(x)) - A(x, dv(x)), du(x) - dv(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Let $F \in L^{p'}(\Omega; \Lambda^{k+1})$ and $f \in L^{p'}(\Omega; \Lambda^k)$. Then there exists a solution $\omega \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ satisfying,

$$\int_{\Omega} \langle A(x, d\omega), d\theta \rangle + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0, \quad (7.12)$$

for all $\theta \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$. In other words, $\omega \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ is a weak solution to the following boundary value problem

$$\begin{cases} \delta(A(x, d\omega)) = f + \delta F & \text{in } \Omega, \\ \delta\omega = 0 & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof We start by defining the operator $a : W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \times W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ by,

$$a(u, v) = \int_{\Omega} \langle A(x, du), dv \rangle.$$

Clearly, $a : W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \times W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ is linear in the second variable but nonlinear in the first. Our plan is to use Minty-Browder theory of monotone operators (cf. theorem 3 in [18]). First note that the operator a is separately continuous in both variables in view of the following estimates,

$$|a(u, v)| = \left| \int_{\Omega} \langle A(x, du), dv \rangle \right| \leq \left(\int_{\Omega} |A(x, du)|^{p'} \right)^{\frac{1}{p'}} \|dv\|_{L^p} \leq (c_3 \|du\|_{L^p}^p + c_4)^{\frac{1}{p'}} \|dv\|_{L^p}$$

since Ω is bounded, i.e $|\Omega| < \infty$. So we need to check coercivity and monotonicity.

Coercivity We have,

$$|a(u, u)| = \left| \int_{\Omega} \langle A(x, du), du \rangle \right| \geq c_2 \|du\|_{L^p}^p - c_5.$$

This proves coercivity since $p > 1$.

Monotonicity

To prove monotonicity of the of the operator a we need to show,

$$a(u, u - v) - a(v, u - v) \geq 0 \quad \text{for all } u, v \in W_{\delta,T}^{d,p}(\Omega, \Lambda^k).$$

But this follows from (N3). This proves monotonicity.

Existence We have shown that the function $a : W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \times W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ is monotone and coercive on the reflexive Banach space $W_{\delta,T}^{d,p}(\Omega, \Lambda^k)$. Since for any $F \in L^{p'}(\Omega, \Lambda^{k+1})$ and any $f \in L^{p'}(\Omega, \Lambda^k)$, where p' is the Hölder conjugate exponent of p , the map $\theta \mapsto -\int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle$ defines a continuous linear functional on $W_{\delta,T}^{d,p}(\Omega, \Lambda^k)$, by theorem 3 in [18], we obtain the existence of $\omega \in W_{\delta,T}^{d,p}(\Omega, \Lambda^k)$ such that,

$$a(\omega, \theta) = -\int_{\Omega} \langle f, \theta \rangle + \int_{\Omega} \langle F, d\theta \rangle \quad \text{for all } \theta \in W_{\delta,T}^{d,p}(\Omega, \Lambda^k).$$

But this implies,

$$a(\omega, \theta) + \int_{\Omega} \langle f, \theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0 \quad \text{for all } \theta \in W_T^{d,2,p}(\Omega, \Lambda^k).$$

This completes the proof. ■

Existence in $W_T^{d,p}$

Theorem 7.13 *Let $1 \leq k \leq n-1$ be an integer and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $A : \Omega \times \Lambda^{k+1} \rightarrow \Lambda^{k+1}$ be a measurable map such that*

(N1) *There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$|A(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) *There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$\langle A(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) *For every $u, v \in W^{d,p}(\Omega, \Lambda^k)$,*

$$\langle A(x, du(x)) - A(x, dv(x)), du(x) - dv(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Let $F \in L^p(\Omega; \Lambda^{k+1})$. Then there exists a solution $\omega \in W_{\delta, T}^{d,p}(\Omega; \Lambda^k)$ satisfying,

$$\int_{\Omega} \langle A(x, d\omega), d\phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = 0, \quad (7.13)$$

for all $\phi \in W_T^{d,p}(\Omega; \Lambda^k)$. In other words, $\omega \in W_{\delta, T}^{d,p}(\Omega; \Lambda^k)$ is a weak solution to the following boundary value problem

$$\begin{cases} \delta(A(x, d\omega)) = \delta F & \text{in } \Omega, \\ \delta\omega = 0 & \text{in } \Omega, \\ \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof By theorem 7.12, there exists a solution $\omega \in W_{\delta, T}^{d,p}(\Omega; \Lambda^k)$ which satisfies

$$\int_{\Omega} \langle A(x, d\omega); d\theta \rangle - \int_{\Omega} \langle F; d\theta \rangle = 0, \quad (7.14)$$

for all $\theta \in W_{\delta, T}^{d,p}(\Omega; \Lambda^k)$. Now, by the decomposition $W_T^{d,p}(\Omega; \Lambda^k) = W_{\delta, T}^{d,p}(\Omega; \Lambda^k) \oplus dW_0^{1,p}(\Omega; \Lambda^k)$, for any $\phi \in W_T^{d,p}(\Omega; \Lambda^k)$, there exist $\theta \in W_{\delta, T}^{d,p}(\Omega; \Lambda^k)$ and $\psi \in W_0^{1,p}(\Omega; \Lambda^k)$ such that

$$\phi = \theta + d\psi.$$

Thus, we have,

$$\int_{\Omega} \langle A(x, d\omega), d\phi \rangle - \int_{\Omega} \langle F, d\phi \rangle = \int_{\Omega} \langle A(x, d\omega), d\theta \rangle - \int_{\Omega} \langle F, d\theta \rangle = 0.$$

This proves the theorem. ■

7.4.2 Main theorems

Theorem 7.14 *Let $1 \leq k \leq n - 1$ be an integer and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. Let $A : \Omega \times \Lambda^{k+1} \rightarrow \Lambda^{k+1}$ be a measurable map such that*

(N1) *There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$|A(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) *There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^{k+1}$,*

$$\langle A(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) *For every $u, v \in W^{d,p}(\Omega, \Lambda^k)$,*

$$\langle A(x, du(x)) - A(x, dv(x)), du(x) - dv(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Then for any $\omega_0 \in W^{d,p}(\Omega, \Lambda^k)$ and any $F \in L^{p'}(\Omega; \Lambda^{k+1})$, there exists a weak solution $\omega \in W^{1,p}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} \delta(A(x, d\omega)) = \delta F & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_D)$$

Proof The idea of the proof is very similar to what we did above. We define the map $\tilde{a} : W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \times W_{\delta,T}^{d,p}(\Omega; \Lambda^k) \rightarrow \mathbb{R}$ by,

$$\tilde{a}(u, v) = \int_{\Omega} \langle A(x, du + d\omega_0), dv \rangle.$$

Proceeding exactly as in theorem 7.12, we can show that this map satisfies all the hypothesis of the monotone operator theory. Hence, we can deduce, using the same line of argument as in theorem 7.12 and theorem 7.13 that there exists a weak solution $\bar{\omega} \in W_{\delta,T}^{d,p}(\Omega; \Lambda^k)$ to the following boundary value problem

$$\begin{cases} \delta(A(x, d\bar{\omega} + d\omega_0)) = \delta F & \text{in } \Omega, \\ \delta\bar{\omega} = 0 & \text{in } \Omega, \\ \nu \wedge \bar{\omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

The rest of the proof is very similar to the linear case. Note that $\bar{\omega} \in W^{1,p}(\Omega; \Lambda^k)$ because of the embedding. Now since $\nu \wedge (-\bar{\omega}) = 0$ on $\partial\Omega$, we can find $v \in W^{2,p}(\Omega, \Lambda^{k-1})$ (cf. lemma 8.11 in [21]) such that $dv = -\bar{\omega}$ on $\partial\Omega$. Then setting $\omega = \omega_0 + \bar{\omega} + dv \in W^{1,p}(\Omega, \Lambda^k)$, we have,

$$\delta(A(x, d\omega)) = \delta(A(x, d\omega_0 + d\bar{\omega} + ddv)) = \delta(A(x, d\omega_0) + d\bar{\omega}) = \delta F \quad \text{in } \Omega.$$

Also, since $dv = -\bar{\omega}$ on $\partial\Omega$, we have $\omega = \omega_0$ on $\partial\Omega$. This finishes the proof. ■

Starting with the space $W_{d,N}^{\delta,p}(\Omega; \Lambda^k)$ and using the dual decomposition

$$W_N^{\delta,p}(\Omega; \Lambda^k) = W_{d,N}^{\delta,p}(\Omega; \Lambda^k) \oplus \delta W_0^{1,p}(\Omega; \Lambda^k),$$

we obtain, in the same way, the corresponding dual theorems. We state the theorems below and omit the proof.

Theorem 7.15 *Let $1 \leq k \leq n-1$ be an integer and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $A : \Omega \times \Lambda^{k-1} \rightarrow \Lambda^{k-1}$ be a measurable map such that*

(N1) *There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^{k-1}$,*

$$|A(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) *There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^{k-1}$,*

$$\langle A(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) *For every $u, v \in W^{\delta,p}(\Omega, \Lambda^k)$,*

$$\langle A(x, \delta u(x)) - A(x, \delta v(x)), \delta u(x) - \delta v(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Let $F \in L^{p'}(\Omega; \Lambda^{k-1})$. Then there exists a solution $\omega \in W_{d,N}^{\delta,p}(\Omega; \Lambda^k)$ satisfying,

$$\int_{\Omega} \langle A(x, \delta \omega), \delta \phi \rangle - \int_{\Omega} \langle F, \delta \phi \rangle = 0, \quad (7.15)$$

for all $\phi \in W_N^{\delta,p}(\Omega; \Lambda^k)$. In other words, $\omega \in W_{d,N}^{\delta,p}(\Omega; \Lambda^k)$ is a weak solution to the following boundary value problem

$$\begin{cases} d(A(x, \delta \omega)) = dF \text{ in } \Omega, \\ d\omega = 0 \text{ in } \Omega, \\ \nu \lrcorner \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

Theorem 7.16 *Let $1 \leq k \leq n-1$ be an integer and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open contractible set and let ν be the outward unit normal to the boundary $\partial\Omega$. $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and contractible. Let $A : \Omega \times \Lambda^{k-1} \rightarrow \Lambda^{k-1}$ be a measurable map such that*

(N1) *There exists a constant $c_1 > 0$ such that for every $\xi \in \Lambda^{k-1}$,*

$$|A(x, \xi)| \leq c_1 (|\xi|^{p-1} + 1) \quad \text{for a.e } x \in \Omega.$$

(N2) *There exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda^{k-1}$,*

$$\langle A(x, \xi), \xi \rangle \geq c_2 (|\xi|^p - 1) \quad \text{for a.e } x \in \Omega.$$

(N3) For every $u, v \in W^{\delta,p}(\Omega, \Lambda^k)$,

$$\langle A(x, \delta u(x)) - A(x, \delta v(x)), \delta u(x) - \delta v(x) \rangle \geq 0 \quad \text{for a.e } x \in \Omega.$$

Then for any $\omega_0 \in W^{\delta,p}(\Omega, \Lambda^k)$ and any $F \in L^{p'}(\Omega; \Lambda^{k-1})$, there exists a weak solution $\omega \in W^{1,p}(\Omega, \Lambda^k)$ to the following boundary value problem,

$$\begin{cases} d(A(x, \delta \omega)) = dF \text{ in } \Omega, \\ \omega = \omega_0 \text{ on } \partial\Omega. \end{cases} \quad (\mathcal{P}_{D,\delta})$$

7.4.3 Remark about regularity

We end this thesis with a few remarks about the regularity of weak solutions to an important special case of the quasilinear boundary value problems discussed above. This is the case when $A(x, \xi) = \varrho(|\xi|^2)\xi$ with the function $\varrho : \mathbb{R} \rightarrow \mathbb{R}$ satisfying some structural hypothesis. A typical and the most important example of such a ϱ , of course, is given by $\varrho(|\xi|^2) = |\xi|^{\frac{p-2}{2}}$, when the system generalizes the p -Laplace operator to differential forms. In this case, if F is 0, i.e the homogeneous case, interior $C^{1,\alpha}$ regularity of the solution is implicitly contained in [70]. Again when F is 0, the $C^{1,\alpha}$ regularity results up to the boundary for the system

$$\begin{cases} \delta(\varrho(|d\omega|^2)d\omega) = 0 \text{ in } \Omega, \\ \delta\omega = 0 \text{ in } \Omega, \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega. \end{cases}$$

can also be deduced from the results obtained by Hamburger in [35] for ϱ -harmonic Dirichlet k -forms with the same assumptions on ϱ (see also Beck-Stroffolini [14] for a partial regularity result). However, there is so far no regularity results for non-zero F , though it seems possible to obtain some regularity results even in this case.

Appendix A

Notations

We gather here the notations which we will use throughout this thesis. For more details on exterior algebra and differential forms see [21] and for the notions of convexity used in the calculus of variations see [25] .

1. Let k be a nonegative integer and n be a positive integer.
 - We write $\Lambda^k(\mathbb{R}^n)$ (or simply Λ^k) to denote the vector space of all alternating k -linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}} \rightarrow \mathbb{R}$. For $k = 0$, we set $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}$.
 - $\wedge, \lrcorner, \langle ; \rangle$ and, respectively, $*$ denote the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator.
 - If $\{e^1, \dots, e^n\}$ is a basis of \mathbb{R}^n , then, identifying Λ^1 with \mathbb{R}^n ,

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of Λ^k . An element $\xi \in \Lambda^k(\mathbb{R}^n)$ will therefore be written as

$$\xi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1 i_2 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} = \sum_{I \in \mathcal{T}^k} \xi_I e^I$$

where

$$\mathcal{T}^k = \left\{ I = (i_1, \dots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \cdots < i_k \leq n \right\}.$$

We shall identify exterior 1-forms with vectors freely and shall refrain from using the musical notation to denote these identifications, in order not to burden our notations further. Also, we shall often write an exterior k -form as a vector in $\mathbb{R}^{\binom{n}{k}}$, when the alternating structure is not important for our concern. In a similar vein, we shall identify $m \times n$ matrices with the space $\mathbb{R}^{m \times n}$.

We adopt the alphabetical order for comparing two multiindices and we do not reserve a specific symbol for this ordering. The usual ordering symbols, when written in the context of multiindices will denote alphabetical ordering. For example, $I = (1, 4) < J = (2, 3)$.

- We write

$$e^{i_1} \wedge \cdots \wedge \widehat{e^{i_s}} \wedge \cdots \wedge e^{i_k} = e^{i_1} \wedge \cdots \wedge e^{i_{s-1}} \wedge e^{i_{s+1}} \wedge \cdots \wedge e^{i_k}.$$

Similarly, $\widehat{}$ placed over a string of indices (or multiindices) will signify the omission of the string under the $\widehat{}$ sign.

2. Let $\xi \in \mathbb{R}^{N \times n}$ be written as

$$\xi = \left(\xi_i^j \right)_{i \in \{1, \dots, n\}}^{j \in \{1, \dots, N\}} = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^N & \cdots & \xi_n^N \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \dots, \xi_n).$$

Let $2 \leq s \leq \min\{n, N\}$. We define the adjugate matrix of index s

$$\text{adj}_s \xi = \left((\text{adj}_s \xi)_I^J \right)_{I \in \mathcal{T}^s(n)}^{J \in \mathcal{T}^s(N)} \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}$$

whose elements are given, for $I = (i_1, \dots, i_s) \in \mathcal{T}^s$ with $1 \leq i_1 < \dots < i_s \leq n$ and $J = (j_1, \dots, j_s) \in \mathcal{T}^s$ with $1 \leq j_1 < \dots < j_s \leq N$, by

$$(\text{adj}_s \xi)_I^J = (\text{adj}_s \xi)_{i_1 \dots i_s}^{j_1 \dots j_s} = \det \begin{pmatrix} \xi_{i_1}^{j_1} & \cdots & \xi_{i_s}^{j_1} \\ \vdots & \ddots & \vdots \\ \xi_{i_1}^{j_s} & \cdots & \xi_{i_s}^{j_s} \end{pmatrix}.$$

3. *Notation for indices:* The following system of notations will be employed throughout.

- (i) Single indices will be written as lower case english letters, multiindices will be written as upper case english letters.
- (ii) Multiindices will always be indexed by superscripts. The use of a subscript while writing a multiindex is reserved for a special purpose. See (vii) below.
- (iii) $\{i_1 i_2 \dots i_r\}$ and $\{i_1, i_2, \dots, i_r\}$ will both represent the string of indices i_1, i_2, \dots, i_r . Similarly, $\{I^1 I^2 \dots I^r\}$ and $\{I^1, I^2, \dots, I^r\}$ will both represent the string of indices obtained by writing out the indices in the indicated order. Unless explicitly mentioned as representing a set, curly braces will represent the string of indices represented by objects inside the braces, rather than the set of such indices.
- (iv) $(i_1 i_2 \dots i_r)$ and (i_1, \dots, i_r) will stand for the permutation of the indices i_1, i_2, \dots, i_r , i.e of the indices contained in the string of indices inside the brackets.
- (v) $[i_1 i_2 \dots i_r]$ will stand for the increasingly ordered string of indices consisting of the indices i_1, i_2, \dots, i_r . However, $[I^1, I^2, \dots, I^r]$ will represent the corresponding string of multiindices I^1, I^2, \dots, I^r , arranged in the increasing alphabetical order.
- (vi) In the spirit of (iii) above the usual setminus sign will be used to denote deletion of the string of indices. For example, the symbol $\{\{i_1, i_2, \dots, i_r\} \setminus \{i_m\}\}$ will be used to represent the string of indices $\{i_1 i_2 \dots i_{m-1} i_{m+1} \dots i_r\}$ and similarly,

the symbol $(\{i_1, i_2, \dots, i_r\} \setminus \{i_m\})$ will stand for the permutation of $r - 1$ indices $(i_1 i_2 \dots i_{m-1} i_{m+1} \dots i_r)$. The same principle will apply for square brackets too.

- (vii) As a rule, multiindices will be indexed by superscripts only and a single index as a subscript in a multiindex will stand for the multiindex obtained from the multiindex, denoted by the same symbol without the subscript by removing the index in the subscript. For example, I_j^i will represent $[I^i \setminus \{j\}]$.
- (viii) The symbol $(J; I)$, where $J = \{j_1 j_2 \dots j_s\}$ is a string of s single indices, not necessarily ordered and $I = \{I^1 I^2 \dots I^s\}$ is a string of s multiindices, $I^1, I^2, \dots, I^s \in \mathcal{T}^{(k-1)s}$, not necessarily alphabetically ordered, will be reserved to denote the string $\{j_1 I^1 j_2 I^2 \dots j_s I^s\}$. Note that the case $k = 2$, when these I^1, I^2, \dots, I^s are single indices rather than multiindices is also included ¹.
- (ix) In the same spirit, $\{.\}$, $(.)$ and $[.]$ will always represent respectively the string, the permutation and the ordered string of indices corresponding to the string of indices represented by the objects inside the curly braces, the brackets and the square brackets respectively.
- (x) The abovementioned system of notations will be in force even when representing indices as subscripts of superscripts of different objects.

4. *Notation for sum:* We shall also be employing some convention for abbreviation of sums.

- Let $I \in \mathcal{T}^{ks}$ be a multiindex, where $1 \leq k \leq n$ and $1 \leq s \leq [\frac{n}{k}]$ are both integers. Then we shall employ the shorthand \sum_s^I to stand for

$$\sum_{\substack{J=\{j_1 j_2 \dots j_s\}=[j_1 j_2 \dots j_s], \\ \tilde{I}=\{I^1 I^2 \dots I^s\}=[I^1, I^2, \dots, I^s] \\ J \cup \tilde{I} = I}} .$$

In other words, the symbol \sum_s^I will stand for the sum running over all possible choices of s single indices and s $k - 1$ -multiindices such that their union is I . Note that it is only the choice that matters, not the order. Since once we have chosen s single indices, our ordering fixes the unique way of naming them and similarly for the multiindices. Writing in a more detailed and explicit manner, we can also write this sum as,

$$\sum_{\substack{j_l \in I, \ I^l \in \mathcal{T}^{k-1} \text{ and } I^l \subset I \text{ for all } 1 \leq l \leq s, \\ j_l \cap j_m = \emptyset, \ I^l \cap I^m = \emptyset \text{ for all } 1 \leq l < m \leq s, \\ j_1 < j_2 < \dots < j_s, \ I^1 < I^2 < \dots < I^s.}}$$

- The symbols like $\sum_s^{I \setminus I'}$, $\sum_s^{[I \setminus I']}$ and \sum_2^I are to be interpreted in the same spirit as above.

5. *Multiindex notation:* We shall use multiindices quite frequently.

¹This is a rather non-standard notation, but nonetheless is extremely useful for our analysis.

Let $\mathbf{k} = (k_1, \dots, k_m)$ where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$. We denote

$$\Lambda^{\mathbf{k}} = \prod_{i=1}^m \Lambda^{k_i}(\mathbb{R}^n).$$

Likewise, for any integer r ,

$$\Lambda^{\mathbf{k}+\mathbf{r}} := \prod_{i=1}^m \Lambda^{k_i+r}(\mathbb{R}^n)$$

for any $r \in \mathbb{Z} \setminus \{0\}$.

We shall denote elements of $\Lambda^{\mathbf{k}}$ by boldface greek letters. For example, we shall write,

$$\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \Lambda^{\mathbf{k}} \quad \text{and} \quad |\boldsymbol{\xi}| := \left(\sum_{i=1}^m |\xi_i|^2 \right)^{\frac{1}{2}}.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{\mathbb{N} \cup \{0\}\}^m$ be a multiindex, in the usual multiindex notations, with $0 \leq \alpha_i \leq \left\lfloor \frac{n}{k_i} \right\rfloor$ for all $1 \leq i \leq m$. We denote $|\boldsymbol{\alpha}| = \sum_{i=1}^m \alpha_i$ and $|\mathbf{k}\boldsymbol{\alpha}| = \sum_{i=1}^m k_i \alpha_i$.

We define, for $|\mathbf{k}\boldsymbol{\alpha}| < n$,

$$\boldsymbol{\xi}^{\boldsymbol{\alpha}} := \xi_1^{\alpha_1} \wedge \dots \wedge \xi_m^{\alpha_m},$$

where the powers on the right hand side represent wedge powers (e.g $\xi_1^2 = \xi_1 \wedge \xi_1$). and ,

$$(*\boldsymbol{\xi})^{\boldsymbol{\alpha}} := (*\xi_1)^{\alpha_1} \wedge \dots \wedge (*\xi_m)^{\alpha_m},$$

where the $*$ represents the Hodge star operator. $*\boldsymbol{\xi}$ is also defined similarly, i.e $*\boldsymbol{\xi} = *\xi_1 \wedge \dots \wedge *\xi_m$.

Also, for any integer $1 \leq s \leq n$, $T_s(\boldsymbol{\xi})$ stands for the vector with components $\boldsymbol{\xi}^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ varies over all possible choices such that $|\boldsymbol{\alpha}| = s$.

6. *Flip*: We shall be employing some particular permutations often.

Definition A.1 (1-flip) Let $s \geq 1$, let $J \in \mathcal{T}^s$, $I \in \mathcal{T}^s$ be written as, $J = \{j_1 \dots j_s\}$, $I = \{i_1 \dots i_s\}$ with $J \cap I = \emptyset$. Let $\tilde{J} \in \mathcal{T}^s$, $\tilde{I} \in \mathcal{T}^l$. We say that (\tilde{J}, \tilde{I}) is obtained from (J, I) by a 1-flip interchanging j_p with i_m , for some $1 \leq p \leq s$, $1 \leq m \leq l$, if

$$\tilde{J} = [j_1 \dots j_{p-1} i_m j_{p+1} \dots j_l] \quad \text{and} \quad \tilde{I} = [i_1 \dots i_{m-1} j_p i_{m+1} \dots i_s].$$

Definition A.2 (k-flip) Let $s \geq 2$, $k \geq 2$ and $I = \{I^1 \dots I^s\} = [I^1, \dots, I^s]$, where $I^1, \dots, I^s \in \mathcal{T}^k$, $I^r = \{i_{r1}^r, \dots, i_{rk}^r\}$ for all $1 \leq r \leq s$. We say that \tilde{I} is obtained from I by a k-flip if there exist integers $1 \leq q_1 < q_2 \leq s$ and $1 \leq r_1, r_2 \leq k$ such that,

$$\tilde{I} = [I^1, \dots, I^{q_1-1}, \tilde{I}^{q_1}, I^{q_1+1}, \dots, I^{q_2-1}, \tilde{I}^{q_2}, I^{q_2+1}, \dots, I^s]$$

where

$$\tilde{I}^{q_1} = [i_{r_1-1}^{q_1}, \dots, i_{r_1-1}^{q_1}, i_{r_2}^{q_2}, i_{r_1+1}^{q_1}, \dots, i_k^{q_1}] \quad \text{and} \quad \tilde{I}^{q_2} = [i_1^{q_2}, \dots, i_{r_2-1}^{q_2}, i_{r_1}^{q_1}, i_{r_2+1}^{q_2}, \dots, i_k^{q_2}].$$

Definition A.3 (alternating k -flip) Let $s \geq 1$, $k \geq 2$. Let $J \in \mathcal{T}^s$, $J = \{j_1 \dots j_s\}$, $I = \{I^1 \dots I^s\} = [I^1, \dots, I^s]$, where $I^1, \dots, I^s \in \mathcal{T}^k$, $I^r = \{i_1^r, \dots, i_k^r\}$ for all $1 \leq r \leq s$ and $J \cap I = \emptyset$. We say that (\tilde{J}, \tilde{I}) is obtained from (J, I) by an alternating k -flip if there exist integers $1 \leq m, p \leq s$ and $1 \leq q \leq k$ such that,

$$\tilde{J} = [j_1 \dots j_{p-1} i_q^m j_{p+1} \dots j_s],$$

and

$$\tilde{I} = [I^1, \dots, I^{m-1}, [i_1^r \dots i_{q-1}^r j_p i_{q+1}^r \dots i_k^r], I^{m+1}, \dots, I^s].$$

Note that a k -flip can be seen as a permutation in an obvious way.

Appendix B

Function Spaces of Differential Forms

Definition B.1 (Differential form) Let $0 \leq k \leq n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. A differential k -form ω is a measurable function $\omega : \Omega \rightarrow \Lambda^k$. We write $\omega \in \mathcal{M}(\Omega; \Lambda^k)$.

B.1 Usual Function Spaces

Definition B.2 (Lebesgue spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We denote by $L^p(\Omega; \Lambda^k)$ the space of all (measurable) differential k -forms $\omega \in \mathcal{M}(\Omega; \Lambda^k)$ for which

$$\begin{aligned} \|\omega\|_{L^p(\Omega; \Lambda^k)} &= \left(\int_{\Omega} |\omega|^p \right)^{\frac{1}{p}} < \infty, & \text{if } 1 \leq p < \infty \\ \|\omega\|_{L^\infty(\Omega; \Lambda^k)} &= \operatorname{ess\,sup}_{\Omega} |\omega| < \infty, & \text{if } p = \infty, \end{aligned}$$

with the abovementioned norms.

Definition B.3 (C^r spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $r \geq 0$ be an integer.

1. We denote by $C^r(\Omega; \Lambda^k)$ the space of all differential k -forms for which all partial derivatives $D^\alpha \omega^I$ for every $I \in \mathcal{T}^k$ and every $0 \leq |\alpha| \leq r$ are continuous. When $r = 0$, we often employ the notation $C(\Omega) := C^0(\Omega)$.
2. $C^r(\overline{\Omega}; \Lambda^k)$ denotes the space of $C^r(\Omega; \Lambda^k)$ forms whose derivatives up to order r can be extended continuously to $\overline{\Omega}$. It is endowed with the norm

$$\|\omega\|_{C^r(\overline{\Omega}; \Lambda^k)} := \max_{\substack{0 \leq |\alpha| \leq r, \\ I \in \mathcal{T}^k}} \sup_{x \in \overline{\Omega}} \|D^\alpha \omega^I(x)\|.$$

3. $C_c(\Omega; \Lambda^k) := \{\omega \in C(\Omega; \Lambda^k) : \operatorname{supp} \omega \subset \Omega \text{ is relatively compact}\}.$
4. $C_c^r(\Omega; \Lambda^k) := C^r(\Omega; \Lambda^k) \cap C_c(\Omega; \Lambda^k).$
5. $C^\infty(\Omega; \Lambda^k) := \bigcap_{r=0}^{\infty} C^r(\Omega; \Lambda^k)$ and $C_c^\infty(\Omega; \Lambda^k) := C_c(\Omega; \Lambda^k) \cap C^\infty(\Omega; \Lambda^k).$

6. $\text{Aff}(\overline{\Omega}; \mathbb{R}^N)$ stands for the set of affine functions, i.e if $u \in \text{Aff}(\Omega; \mathbb{R}^N)$ means there exists $\xi \in \mathbb{R}^{N \times n}$ such that $\nabla u(x) = \xi$ for every $x \in \overline{\Omega}$.
7. We say $u \in C_{\text{piece}}^r(\overline{\Omega}; \mathbb{R}^N)$ if $u \in C^{r-1}(\overline{\Omega}; \mathbb{R}^N)$ and ∇u is piecewise continuous, meaning that there exists a partition of Ω into a countable union of disjoint open sets $\Omega_k \subset \Omega$, i.e $\Omega_h \cap \Omega_k = \emptyset$ if $h, k \in \mathbb{N}, h \neq k$ and $|\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k| = 0$ so that $\nabla u \in C(\overline{\Omega_k}; \mathbb{R}^{N \times n^r})$. $\text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^N)$ stands for the subset of $C_{\text{piece}}^1(\overline{\Omega}; \mathbb{R}^N)$ such that ∇u is piecewise constant.

Definition B.4 (Hölder spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $r \geq 0$ be an integer and $0 < \alpha \leq 1$. For $u : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we define the α -Hölder seminorm as,

$$[u]_{\alpha, D} := \sup_{\substack{x, y \in \Omega, \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}.$$

Now we define the different Hölder spaces in the following way:

1. We denote by $C^{r, \alpha}(\Omega; \Lambda^k)$ the space of all $\omega \in C^r(\Omega; \Lambda^k)$ for which $[D^\beta \omega^I]_{\alpha, K} < \infty$ for every $I \in \mathcal{T}^k$ and every $0 \leq |\beta| \leq r$ for every compact $K \subset \Omega$. When $r = 0$, we often employ the notation $C^{0, \alpha}(\Omega; \Lambda^k) := C^\alpha(\Omega; \Lambda^k)$.
2. We denote by $C^{r, \alpha}(\overline{\Omega}; \Lambda^k)$ the space of all $\omega \in C^r(\overline{\Omega}; \Lambda^k)$ for which $[D^\beta \omega^I]_{\alpha, \overline{\Omega}} < \infty$ for every $I \in \mathcal{T}^k$ and every $0 \leq |\beta| \leq r$. It is endowed with the norm

$$\|\omega\|_{C^{r, \alpha}(\overline{\Omega}; \Lambda^k)} := \|\omega\|_{C^r(\overline{\Omega}; \Lambda^k)} + \max_{\substack{0 \leq |\beta| \leq r, \\ I \in \mathcal{T}^k}} [D^\beta \omega^I]_{\alpha, \overline{\Omega}}.$$

3. By abuse of notation, we often write $C^r(\Omega; \Lambda^k) = C^{r, 0}(\Omega; \Lambda^k)$.
4. $C^{r, 1}(\overline{\Omega}; \Lambda^k)$ is identified with all $\omega \in C^{r-1}(\overline{\Omega}; \Lambda^k)$ such that $D^\beta \omega^I$ is Lipschitz continuous for every $I \in \mathcal{T}^k$ and every $|\beta| = r$.

Definition B.5 (Sobolev spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. Let $r \geq 0$ be an integer.

1. We define $W^{r, p}(\Omega; \Lambda^k)$ to be the spaces of differential k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$ and $D^\alpha \omega^I \in L^p(\Omega)$ for every $I \in \mathcal{T}^k$ and every $0 \leq |\alpha| \leq r$, where D^α is the weak derivative in the usual multiindex notation. It is endowed with the norm

$$\begin{aligned} \|\omega\|_{W^{r, p}(\Omega; \Lambda^k)} &:= \sum_{0 \leq |\alpha| \leq r} \sum_{I \in \mathcal{T}^k} \|D^\alpha \omega^I\|_{L^p(\Omega)} \quad \text{if } 1 \leq p < \infty \\ \|\omega\|_{W^{r, \infty}(\Omega; \Lambda^k)} &:= \max_{\substack{0 \leq |\alpha| \leq r, \\ I \in \mathcal{T}^k}} \|D^\alpha \omega^I\|_{L^\infty(\Omega)} \quad \text{if } p = \infty. \end{aligned}$$

2. If $1 \leq p < \infty$, the space $W_0^{r, p}(\Omega; \Lambda^k)$ is defined as the closure of $C_c^\infty(\Omega; \Lambda^k)$ in $W^{r, p}(\Omega; \Lambda^k)$.
3. We define $W_0^{r, \infty}(\Omega; \Lambda^k) := W^{r, \infty}(\Omega; \Lambda^k) \cap W_0^{r, 1}(\Omega; \Lambda^k)$.
4. For $r = 0$, the spaces $W^{r, p}(\Omega; \Lambda^k)$ coincide with $L^p(\Omega; \Lambda^k)$ with equivalent norms.

We now list the a few well-known results about these spaces.

Proposition B.6 (Sobolev Embeddings) *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary.*

- *If $1 \leq p < n$, then*

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow L^q(\Omega; \Lambda^k)$$

for every $1 \leq q \leq p^$, where p^* is the Sobolev conjugate exponent of p , defined by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \text{ i.e. } p^* = \frac{np}{n-p}.$$

More precisely, for every $1 \leq q \leq p^$ there exists a constant $c = c(\Omega, p, q)$ such that*

$$\|\omega\|_{L^q} \leq c \|\omega\|_{W^{1,p}}$$

for every $\omega \in W^{1,p}(\Omega; \Lambda^k)$.

- *If $p = n$, then*

$$W^{1,n}(\Omega; \Lambda^k) \hookrightarrow L^q(\Omega; \Lambda^k)$$

for every $1 \leq q < \infty$. More precisely, for every $1 \leq q < \infty$ there exists a constant $c = c(\Omega, p, q)$ such that

$$\|\omega\|_{L^q} \leq c \|\omega\|_{W^{1,n}}$$

for every $\omega \in W^{1,n}(\Omega; \Lambda^k)$.

- *If $p > n$, then*

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow C^{0,\alpha}(\overline{\Omega}; \Lambda^k)$$

for every $0 \leq \alpha \leq 1 - \frac{n}{p}$. In particular, there exists a constant $c = c(\Omega, p, q)$ such that

$$\|\omega\|_{L^\infty} \leq c \|\omega\|_{W^{1,p}}.$$

Proposition B.7 (Rellich-Kondrachov) *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary.*

- *If $1 \leq p < n$, then the embedding*

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow L^q(\Omega; \Lambda^k)$$

is compact for every $1 \leq q < p^$, where p^* is the Sobolev conjugate exponent of p , defined by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \text{ i.e. } p^* = \frac{np}{n-p}.$$

- *If $p = n$, then the embedding*

$$W^{1,n}(\Omega; \Lambda^k) \hookrightarrow L^q(\Omega; \Lambda^k)$$

is compact for every $1 \leq q < \infty$.

- If $p > n$, then the embedding

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow C^{0,\alpha}(\overline{\Omega}; \Lambda^k)$$

is compact for every $0 \leq \alpha < 1 - \frac{n}{p}$.

In particular, the embedding

$$W^{1,p}(\Omega; \Lambda^k) \hookrightarrow L^p(\Omega; \Lambda^k)$$

is compact for every $1 \leq p \leq \infty$.

B.2 Special function spaces

Apart from the usual function spaces defined above, we shall be using several function spaces which are particularly well suited for working with differential forms. First, we define the differential operators we are concerned with.

Definition B.8 (Exterior derivative) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $\omega \in L^1_{\text{loc}}(\Omega; \Lambda^k)$. A differential $(k+1)$ -form $\varphi \in L^1_{\text{loc}}(\Omega; \Lambda^{k+1})$ is called the exterior derivative of ω , denoted by $d\omega$, if

$$\int_{\Omega} \eta \wedge \varphi = (-1)^{n-k} \int_{\Omega} d\eta \wedge \omega,$$

for all $\eta \in C_0^\infty(\Omega; \Lambda^{n-k-1})$.

The formal adjoint of this operator is also very important for our purposes.

Definition B.9 (Hodge codifferential) Let $1 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be open and let $\omega \in L^1_{\text{loc}}(\Omega; \Lambda^k)$ be such that $d\omega$ exists. Then, the Hodge codifferential of ω is a $(k-1)$ -form $\delta\omega \in L^1_{\text{loc}}(\Omega; \Lambda^{k-1})$ defined as

$$\delta\omega := (-1)^{nk+1} * d * \omega.$$

Since differentiation on forms occurs only via operators d and δ , the following spaces are of crucial importance. See [40] for more detail.

Definition B.10 (Partial Sobolev spaces) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We define $W^{d,p}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$ and $d\omega \in L^p(\Omega; \Lambda^{k+1})$. It is endowed with the norm

$$\|\omega\|_{d,p} := \|\omega\|_p + \|d\omega\|_p, \text{ for all } \omega \in W^{d,p}(\Omega; \Lambda^k).$$

Similarly, for $1 \leq k \leq n$, we define $W^{\delta,p}(\Omega; \Lambda^k)$ as the space of differential k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\delta,p} := \|\omega\|_p + \|\delta\omega\|_p, \text{ for all } \omega \in W^{\delta,p}(\Omega; \Lambda^k).$$

Definition B.11 (Partial Sobolev spaces of (p, q) type) Let $0 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p, q \leq \infty$. We define $W^{d,p,q}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^q(\Omega; \Lambda^k)$ and $d\omega \in L^p(\Omega; \Lambda^{k+1})$, endowed with the norm

$$\|\omega\|_{d,p,q} := (\|\omega\|_q^2 + \|d\omega\|_p^2)^{\frac{1}{2}}, \text{ for all } \omega \in W^{d,p,q}(\Omega; \Lambda^k).$$

Similarly, for $1 \leq k \leq n$, we define $W^{\delta,p,q}(\Omega; \Lambda^k)$ to be the space of differential k -forms such that $\omega \in L^q(\Omega; \Lambda^k)$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\delta,p} := (\|\omega\|_q^2 + \|\delta\omega\|_p^2)^{\frac{1}{2}}, \text{ for all } \omega \in W^{\delta,p,q}(\Omega; \Lambda^k).$$

Definition B.12 (Total Sobolev spaces) Let $1 \leq k \leq n-1$, let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. We define $\mathcal{L}^{1,p}(\Omega; \Lambda^k)$ to be the space of k -forms such that $\omega \in L^p(\Omega; \Lambda^k)$, $d\omega \in L^p(\Omega; \Lambda^{k+1})$ and $\delta\omega \in L^p(\Omega; \Lambda^{k-1})$, equipped with the norm

$$\|\omega\|_{\mathcal{L}^{1,p}} := \|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p, \text{ for all } \omega \in \mathcal{L}^{1,p}(\Omega; \Lambda^k).$$

For the spaces mentioned above, although the usual notion of trace does not always make sense, one can define partial traces on these spaces. We denote by $\nu \wedge \omega$ and $\nu \lrcorner \omega$ as the tangential and normal trace, respectively, of a function ω , when they are defined. The subspaces with zero tangential and normal traces are important too.

Definition B.13 Let $0 \leq k \leq n$, let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $1 \leq p < \infty$. We define

$$\begin{aligned} W_T^{d,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{d,p}(\Omega; \Lambda^k) : \langle d\omega; \phi \rangle = \langle \omega; \delta\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k+1}) \right\}. \\ W_N^{\delta,p}(\Omega; \Lambda^k) &:= \left\{ \omega \in W^{\delta,p}(\Omega; \Lambda^k) : \langle \delta\omega; \phi \rangle = \langle \omega; d\phi \rangle, \text{ for all } \phi \in C^\infty(\overline{\Omega}; \Lambda^{k-1}) \right\}. \end{aligned}$$

Definition B.14 We set

$$W_T^{d,\infty}(\Omega; \Lambda^k) = W^{d,\infty}(\Omega; \Lambda^k) \cap W_T^{d,1}(\Omega; \Lambda^k).$$

We shall also be needing spaces suited to working with several differential forms.

Definition B.15 Let $\mathbf{p} = (p_1, \dots, p_m)$ where $1 \leq p_i \leq \infty$ for all $1 \leq i \leq m$. We define the spaces $L^{\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$ and $W^{1,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$, $W^{d,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}})$ to be the corresponding product spaces. E.g.

$$W^{d,\mathbf{p}}(\Omega, \Lambda^{\mathbf{k}}) := \prod_{i=1}^m W^{d,p_i}(\Omega, \Lambda^{k_i}).$$

They are obviously also endowed with the corresponding product norms. When $p_i = \infty$ for all $1 \leq i \leq m$, we denote the corresponding spaces by L^∞ , $W^{1,\infty}$ etc.

Definition B.16 In the same manner, $\omega^\nu \rightharpoonup \omega$ in $W^{d,\mathbf{p}}(\Omega; \Lambda^{k-1})$ will stand for a shorthand of

$$\omega_i^\nu \rightharpoonup \omega_i \text{ in } W^{d,p_i}(\Omega; \Lambda^{k_i-1}),$$

for all $1 \leq i \leq m$, and $f(\mathbf{d}\omega^\nu) \xrightarrow{*} f(\mathbf{d}\omega)$ in $\mathcal{D}'(\Omega)$ will mean

$$f(d\omega_1^\nu, \dots, d\omega_m^\nu) \xrightarrow{*} f(d\omega_1, \dots, d\omega_m) \text{ in } \mathcal{D}'(\Omega).$$

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Publications

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- Bandyopadhyay S. and Sil S., Exterior convexity and classical Calculus of variations. ESAIM Control Optim. Calc. Var., 22(2):338-354, 2016.
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